# 第二章　在 Schwarzschild 背景中的 Axial 重力微擾

在廣義相對論中，時空的彎曲由 Einstein 方程式描述

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} \qquad (2.1)$$

其中 $g_{\mu\nu}$ 稱為 Riemann 度規張量，由下式定義

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$R_{\mu\nu}$ 是 Ricci 張量，定義為

$$R_{\mu\nu} = \Gamma^{\alpha}{}_{\mu\alpha,\nu} - \Gamma^{\alpha}{}_{\mu\nu,\alpha} + \Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta} - \Gamma^{\beta}{}_{\mu\nu}\Gamma^{\alpha}{}_{\alpha\beta}$$

而 $R$ 是 Ricci 純量，定義為

$$R = g^{\mu\nu} R_{\mu\nu} \qquad (2.2)$$

這裡的 $\Gamma^{\kappa}{}_{\mu\nu}$ 是 Christoffel symbol，定義為

$$\Gamma^{\kappa}{}_{\mu\nu} = \frac{1}{2} g^{\kappa\alpha}\left(g_{\alpha\nu,\mu} + g_{\alpha\mu,\nu} - g_{\mu\nu,\alpha}\right)$$

$T_{\mu\nu}$ 稱為物質的能量動量張量，當 $T_{\mu\nu} = 0$ 時，即為真空，(2.1)式變成

$$R_{\mu\nu} = \frac{1}{2} g_{\mu\nu} R \qquad (2.3)$$

由(2.2)式及(2.3)式可得

$$R = \frac{1}{2} g^{\nu\mu} g_{\mu\nu} R = 2R$$

故得 $R = 0$，(2.3)式變成

$$R_{\mu\nu} = 0$$

為了簡化符號，我們將重力常數 $G$、光速 $c$ 與 Planck 常數 $\hbar$ 皆設為 1。現在考慮一個靜態的真空解，時空度規為 Schwarzschild 黑洞，其形式如下

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

$$\left(g_{\mu\nu}\right) = \begin{pmatrix} -(1-2M/r) & 0 & 0 & 0 \\ 0 & (1-2M/r)^{-1} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{pmatrix} \qquad \mu,\nu = 0,1,2,3$$

$$\left(g^{\mu\nu}\right) = (g^{-1})^{\mu\nu} = \begin{pmatrix} -(1-2M/r)^{-1} & 0 & 0 & 0 \\ 0 & (1-2M/r) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

$r = 2M$ 是 Schwarzschild 黑洞的現象視界（event horizon），$r = 0$ 是 Schwarzschild 黑洞的奇點(singularity)。

## 2.1 線性化重力場方程式

將總度規 $\bar{g}_{\mu\nu}$ 分為背景度規 $g_{\mu\nu}$ 與微擾度規 $h_{\mu\nu}$

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \qquad (2.4)$$

考慮擾動不大的情況下，我們可以只計算 Ricci 張量到 $h$ 的第一階為止

$$h^{\mu\nu} = g^{\mu\alpha} g^{\kappa\nu} h_{\alpha\kappa}$$

$$h^{\mu\alpha} g_{\alpha\nu} = g^{\mu\alpha} h_{\alpha\nu} \qquad (2.5)$$

$$\bar{g}^{\mu\alpha} \bar{g}_{\alpha\nu} = \delta^{\mu}_{\nu} + O(h^2) \qquad (2.6)$$

由(2.4)、(2.5)及(2.6)式可得

$$\bar{g}^{\mu\nu} = g^{\mu\nu} - h^{\mu\nu} + O(h^2)$$

Christoffel symbol

$$
\begin{aligned}
\overline{\Gamma}^{\kappa}{}_{\mu\nu} &= \frac{1}{2}\overline{g}^{\kappa\alpha}\left(\overline{g}_{\alpha\nu,\mu}+\overline{g}_{\alpha\mu,\nu}-\overline{g}_{\mu\nu,\alpha}\right) \\
&= \frac{1}{2}\left[g^{\kappa\alpha}-h^{\kappa\alpha}+O(h^2)\right]\left[\frac{\partial}{\partial x^{\mu}}\left(g_{\alpha\nu}+h_{\alpha\nu}\right)+\frac{\partial}{\partial x^{\nu}}\left(g_{\alpha\mu}+h_{\alpha\mu}\right)-\frac{\partial}{\partial x^{\alpha}}\left(g_{\mu\nu}+h_{\mu\nu}\right)\right] \\
&= \frac{1}{2}\left[g^{\kappa\alpha}-h^{\kappa\alpha}+O(h^2)\right]\left[\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)+\left(h_{\alpha\nu,\mu}+h_{\alpha\mu,\nu}-h_{\mu\nu,\alpha}\right)\right] \\
&= \frac{1}{2}g^{\kappa\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right) \\
&\quad +\frac{1}{2}g^{\kappa\alpha}\left(h_{\alpha\nu,\mu}+h_{\alpha\mu,\nu}-h_{\mu\nu,\alpha}\right)-\frac{1}{2}h^{\kappa\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)+O(h^2) \\
&= \Gamma^{\kappa}{}_{\mu\nu}+\delta\Gamma^{\kappa}{}_{\mu\nu}+O(h^2)
\end{aligned}
$$

Christoffel symbol 的變化

$$
\delta\Gamma^{\kappa}{}_{\mu\nu}=\frac{1}{2}g^{\kappa\alpha}\left(h_{\alpha\nu,\mu}+h_{\alpha\mu,\nu}-h_{\mu\nu,\alpha}\right)-\frac{1}{2}h^{\kappa\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)
$$

由於$\frac{1}{2}h^{\kappa\alpha}\left(g_{\alpha\nu,\mu}+g_{\alpha\mu,\nu}-g_{\mu\nu,\alpha}\right)=g^{\kappa\rho}\Gamma^{\sigma}{}_{\mu\nu}h_{\rho\sigma}$，我們可以把上式寫成

$$
\begin{aligned}
\delta\Gamma^{\kappa}{}_{\mu\nu} &= \frac{1}{2}g^{\kappa\alpha}\left(h_{\alpha\nu,\mu}+h_{\alpha\mu,\nu}-h_{\mu\nu,\alpha}\right)-g^{\kappa\alpha}\Gamma^{\beta}{}_{\mu\nu}h_{\alpha\beta} \\
&= \frac{1}{2}g^{\kappa\alpha}\left(h_{\alpha\nu;\mu}+h_{\alpha\mu;\nu}-h_{\mu\nu;\alpha}\right)
\end{aligned}
$$

Ricci 張量

$$
\begin{aligned}
\overline{R}_{\mu\nu} &= \overline{\Gamma}^{\alpha}{}_{\mu\alpha,\nu}-\overline{\Gamma}^{\alpha}{}_{\mu\nu,\alpha}+\overline{\Gamma}^{\beta}{}_{\mu\alpha}\overline{\Gamma}^{\alpha}{}_{\nu\beta}-\overline{\Gamma}^{\beta}{}_{\mu\nu}\overline{\Gamma}^{\alpha}{}_{\alpha\beta} \\
&= \left[\Gamma^{\alpha}{}_{\mu\alpha,\nu}+\delta\Gamma^{\alpha}{}_{\mu\alpha,\nu}+O(h^2)\right]-\left[\Gamma^{\alpha}{}_{\mu\nu,\alpha}+\delta\Gamma^{\alpha}{}_{\mu\nu,\alpha}+O(h^2)\right] \\
&\quad +\left[\Gamma^{\beta}{}_{\mu\alpha}+\delta\Gamma^{\beta}{}_{\mu\alpha}+O(h^2)\right]\left[\Gamma^{\alpha}{}_{\nu\beta}+\delta\Gamma^{\alpha}{}_{\nu\beta}+O(h^2)\right] \\
&\quad -\left[\Gamma^{\beta}{}_{\mu\nu}+\delta\Gamma^{\beta}{}_{\mu\nu}+O(h^2)\right]\left[\Gamma^{\alpha}{}_{\alpha\beta}+\delta\Gamma^{\alpha}{}_{\alpha\beta}+O(h^2)\right] \\
&= \left(\Gamma^{\alpha}{}_{\mu\alpha,\nu}-\Gamma^{\alpha}{}_{\mu\nu,\alpha}+\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta}-\Gamma^{\beta}{}_{\mu\nu}\Gamma^{\alpha}{}_{\alpha\beta}\right) \\
&\quad +\left(\delta\Gamma^{\alpha}{}_{\mu\alpha,\nu}-\delta\Gamma^{\alpha}{}_{\mu\nu,\alpha}+\delta\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta}+\Gamma^{\beta}{}_{\mu\alpha}\delta\Gamma^{\alpha}{}_{\nu\beta}-\delta\Gamma^{\beta}{}_{\mu\nu}\Gamma^{\alpha}{}_{\alpha\beta}-\Gamma^{\beta}{}_{\mu\nu}\delta\Gamma^{\alpha}{}_{\alpha\beta}\right)+O(h^2) \\
&= R_{\mu\nu}+\delta R_{\mu\nu}+O(h^2)
\end{aligned}
$$

Ricci 張量的微擾

$$
\begin{aligned}
\delta R_{\mu\nu} &= \delta\Gamma^{\alpha}{}_{\mu\alpha,\nu}-\delta\Gamma^{\alpha}{}_{\mu\nu,\alpha}+\delta\Gamma^{\beta}{}_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta}+\Gamma^{\beta}{}_{\mu\alpha}\delta\Gamma^{\alpha}{}_{\nu\beta}-\delta\Gamma^{\beta}{}_{\mu\nu}\Gamma^{\alpha}{}_{\alpha\beta}-\Gamma^{\beta}{}_{\mu\nu}\delta\Gamma^{\alpha}{}_{\alpha\beta} \\
&= \delta\Gamma^{\alpha}{}_{\mu\alpha;\nu}-\delta\Gamma^{\alpha}{}_{\mu\nu;\alpha}
\end{aligned}
\tag{2.7}
$$

在真空中

$$
\overline{R}_{\mu\nu}=R_{\mu\nu}+\delta R_{\mu\nu}=0
$$

故重力場的微擾方程式變成

$$
\delta R_{\mu\nu}=0 \tag{2.8}
$$

## 2.2 Regge-Wheeler 微擾度規

由於背景具有球對稱，因此微擾方程式可以被分成角度變數$\theta$及$\varphi$的部分與徑向變數$r$及時間變數$t$的部分。$h$的不同分量在旋轉變換下有不同的變換性質

$$h=\begin{bmatrix} S & S & V \\ S & S & V \\ V & V & T \end{bmatrix}$$

$S$的部分在旋轉變換下是純量，$V$的部分是向量，$T$的部分是二階張量。$h$的純量分量可以直接用純量球諧函數$Y_{lm}(\theta,\varphi)$表示，

$$Y_{lm}(\theta,\varphi)=\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_l^m(\cos\theta)\exp(im\varphi) \qquad (2.9)$$

其中$Y_{lm}(\theta,\varphi)$滿足

$$\left[\sin^2\theta\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial\varphi^2}+\cos\theta\sin\theta\frac{\partial}{\partial\theta}+\sin^2\theta\, l(l+1)\right]Y_{lm}(\theta,\varphi)=0 \qquad (2.10)$$

$P_l^m(x)$是 associated Legendre 函數，它滿足

$$\frac{d^2}{d\theta^2}P_l^m(\cos\theta)+\frac{\cos\theta}{\sin\theta}\frac{d}{d\theta}P_l^m(\cos\theta)-\frac{m^2}{\sin^2\theta}P_l^m(\cos\theta)=-l(l+1)P_l^m(\cos\theta) \qquad (2.11)$$

亦即純量函數$S_{lm}(\theta,\varphi)=Y_{lm}(\theta,\varphi)$，向量與張量則可以被建構如下

$$\left(\overset{1}{V}_{lm}\right)_a=\left(S_{lm}\right)_{;a}=\frac{\partial}{\partial x^a}Y_{lm}(\theta,\varphi)$$

$$\left(\overset{2}{V}_{lm}\right)_a=\varepsilon_a{}^b\left(S_{lm}\right)_{;b}=\gamma^{bc}\varepsilon_{ac}\frac{\partial}{\partial x^b}Y_{lm}(\theta,\varphi)$$

$$\left(\overset{1}{T}_{lm}\right)_{ab}=\left(S_{lm}\right)_{;ab}$$

$$\left(\overset{2}{T}_{lm}\right)_{ab}=S_{lm}\gamma_{ab}$$

$$\left(\overset{3}{T}_{lm}\right)_{ab}=\frac{1}{2}\left[\varepsilon_a{}^c\left(S_{lm}\right)_{;cb}+\varepsilon_b{}^c\left(S_{lm}\right)_{;ca}\right]$$

其中$a,b,c=2,3$

$$(\gamma_{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \qquad a, b = 2, 3$$

$$(\gamma^{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{1}{\sin^2\theta} \end{pmatrix}$$

$$(\varepsilon_{ab}) = \begin{pmatrix} 0 & -\sin\theta \\ \sin\theta & 0 \end{pmatrix} \qquad a, b = 2, 3$$

現在考慮空間倒置（宇稱），即 $(\theta,\varphi) \to (\pi-\theta, \pi+\varphi)$，由球諧函數的性質可得

$$Y_{lm}(\pi-\theta, \pi+\varphi) = (-1)^l Y_{lm}(\theta,\varphi)$$

簡單的說，一個純量函數在空間倒置下保持不變的稱為偶宇稱，若改變符號，則稱為奇宇稱。在相對論微擾理論，我們可以把這個概念推廣：函數在空間倒置下獲得一個 $(-1)^l$ 因子稱為偶性，若是獲得 $(-1)^{l+1}$ 因子，則稱為奇性。為避免因不一致的用法所引起的混亂，Chandrasekhar 將獲得 $(-1)^l$ 因子稱為 polar 微擾，而將獲得 $(-1)^{l+1}$ 因子稱為 axial 微擾。在此定義之下，我們將可得到向量與張量在空間倒置下的宇稱性質如下

$$\begin{array}{lll} S_{lm} & polar & (-1)^l \\ \overset{1}{V}_{lm} & polar & (-1)^l \\ \overset{2}{V}_{lm} & axial & (-1)^{l+1} \\ \overset{1}{T}_{lm} & polar & (-1)^l \\ \overset{2}{T}_{lm} & polar & (-1)^l \\ \overset{3}{T}_{lm} & axial & (-1)^{l+1} \end{array}$$

由於背景度規在空間倒置下不變，因此我們可將 polar 微擾與 axial 微擾分開研究。首先只考慮 axial 微擾，axial 微擾的純量分量為 0，axial 微擾的向量與張量分量分別是

向量分量：
$$\left(\overset{2}{V}_{lm}\right)_a = \varepsilon_a{}^b (S_{lm})_{;b} = \gamma^{bc}\varepsilon_{ac}\frac{\partial}{\partial x^b} Y_{lm}(\theta,\varphi)$$

$$\overset{2}{V}_{lm} = \left(\overset{2}{V}_{lm}\right)_a dx^a = \left(\overset{2}{V}_{lm}\right)_2 dx^2 + \left(\overset{2}{V}_{lm}\right)_3 dx^3$$
$$= \left(-\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi} Y_{lm}(\theta,\varphi)\right) d\theta + \left(\sin\theta \frac{\partial}{\partial\theta} Y_{lm}(\theta,\varphi)\right) d\varphi$$

$$\left(\overset{3}{T}_{lm}\right)_{ab}=\frac{1}{2}\left[\varepsilon_a{}^c\left(S_{lm}\right)_{;cb}+\varepsilon_b{}^c\left(S_{lm}\right)_{;ca}\right]$$

張量分量：

$$=\frac{1}{2}\left\{\begin{aligned}&\gamma^{cd}\varepsilon_{ad}\left[\frac{\partial^2 Y_{lm}}{\partial x^c\partial x^b}-\frac{\partial Y_{lm}}{\partial x^m}\frac{1}{2}\gamma^{km}\left(\frac{\partial\gamma_{ck}}{\partial x^b}+\frac{\partial\gamma_{bk}}{\partial x^c}-\frac{\partial\gamma_{cb}}{\partial x^k}\right)\right]\\&+\gamma^{cd}\varepsilon_{bd}\left[\frac{\partial^2 Y_{lm}}{\partial x^c\partial x^a}-\frac{\partial Y_{lm}}{\partial x^m}\frac{1}{2}\gamma^{km}\left(\frac{\partial\gamma_{ck}}{\partial x^a}+\frac{\partial\gamma_{ak}}{\partial x^c}-\frac{\partial\gamma_{ca}}{\partial x^k}\right)\right]\end{aligned}\right\}$$

$$\begin{aligned}\overset{3}{T}_{lm}&=\left(\overset{3}{T}_{lm}\right)_{ab}dx^a dx^b\\&=\left(\overset{3}{T}_{lm}\right)_{22}dx^2dx^2+\left(\overset{3}{T}_{lm}\right)_{23}dx^2dx^3+\left(\overset{3}{T}_{lm}\right)_{32}dx^3dx^2+\left(\overset{3}{T}_{lm}\right)_{33}dx^3dx^3\\&=\left(-\frac{1}{2}\frac{1}{\sin\theta}X_{lm}(\theta,\varphi)\right)d\theta d\theta+\left(\frac{1}{2}\sin\theta W_{lm}(\theta,\varphi)\right)d\theta d\varphi\\&\quad+\left(\frac{1}{2}\sin\theta W_{lm}(\theta,\varphi)\right)d\varphi d\theta+\left(\frac{1}{2}\sin\theta X_{lm}(\theta,\varphi)\right)d\varphi d\varphi\end{aligned}$$

其中

$$X_{lm}(\theta,\varphi)=2\left(\frac{\partial}{\partial\theta}\frac{\partial}{\partial\varphi}Y_{lm}-\cot\theta\frac{\partial}{\partial\varphi}Y_{lm}\right)$$

$$W_{lm}(\theta,\varphi)=\left(\frac{\partial^2}{\partial\theta^2}Y_{lm}-\cot\theta\frac{\partial}{\partial\theta}Y_{lm}-\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}Y_{lm}\right)$$

現在可將 axial 微擾度規 $h_{\mu\nu}$ 的一般形式寫成

$$\left(h_{\mu\nu}\right)=\begin{pmatrix}0&0&-h_0(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}&h_0(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\0&0&-h_1(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial\varphi}&h_1(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\ *&*&-\frac{1}{2}h_2(t,r)\frac{1}{\sin\theta}X_{lm}&\frac{1}{2}h_2(t,r)\sin\theta W_{lm}\\ *&*&*&\frac{1}{2}h_2(t,r)\sin\theta X_{lm}\end{pmatrix}$$

其中的「*」符號表示該分量由對稱性決定，$h_0$、$h_1$ 與 $h_2$ 是 $(t,r)$ 的任意函數。Einstein 方程式中的廣義座標變換相當於一種規範變換，因此可以透過座標變換來連接的度規是彼此等價的。考慮如下的無限小（infinitesimal）座標變換

$$x'^{\mu}=x^{\mu}+\eta^{\mu}$$

$$\bar{g}_{\rho\lambda}(x)=\frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\lambda}}\bar{g}'_{\mu\nu}(x')$$
$$=\left(\delta^{\mu}_{\rho}+\frac{\partial\eta^{\mu}}{\partial x^{\rho}}\right)\left(\delta^{\nu}_{\lambda}+\frac{\partial\eta^{\nu}}{\partial x^{\lambda}}\right)\bar{g}'_{\mu\nu}(x')$$
$$=\bar{g}'_{\rho\lambda}(x')+\frac{\partial\eta^{\nu}}{\partial x^{\lambda}}\bar{g}'_{\rho\nu}(x')+\frac{\partial\eta^{\mu}}{\partial x^{\rho}}\bar{g}'_{\mu\lambda}(x')$$
$$=\bar{g}'_{\rho\lambda}(x')+\frac{\partial\eta^{\nu}}{\partial x^{\lambda}}\bar{g}_{\rho\nu}(x)+\frac{\partial\eta^{\mu}}{\partial x^{\rho}}\bar{g}_{\mu\lambda}(x)$$
$$=\bar{g}'_{\rho\lambda}(x')+\frac{\partial\eta^{\nu}}{\partial x^{\lambda}}g_{\rho\nu}(x)+\frac{\partial\eta^{\mu}}{\partial x^{\rho}}g_{\mu\lambda}(x)$$
$$\bar{g}'_{\mu\nu}(x')=\bar{g}_{\mu\nu}(x)-\frac{\partial\eta^{\alpha}}{\partial x^{\nu}}g_{\mu\alpha}(x)-\frac{\partial\eta^{\alpha}}{\partial x^{\mu}}g_{\alpha\nu}(x)$$
$$\eta_{\mu;\nu}+\eta_{\nu;\mu}=\frac{\partial\eta_{\mu}}{\partial x^{\nu}}+\frac{\partial\eta_{\nu}}{\partial x^{\mu}}-2\eta_{\lambda}\Gamma^{\lambda}{}_{\mu\nu}$$
$$=\frac{\partial}{\partial x^{\nu}}\left(g_{\mu\alpha}\eta^{\alpha}\right)+\frac{\partial}{\partial x^{\mu}}\left(g_{\nu\alpha}\eta^{\alpha}\right)-\eta^{\alpha}\delta^{k}_{\alpha}\left(\frac{\partial g_{\mu k}}{\partial x^{\nu}}+\frac{\partial g_{\nu k}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{k}}\right)$$
$$=g_{\mu\alpha}\frac{\partial\eta^{\alpha}}{\partial x^{\nu}}+g_{\nu\alpha}\frac{\partial\eta^{\alpha}}{\partial x^{\mu}}+\eta^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$
$$\bar{g}'_{\mu\nu}(x')=\bar{g}_{\mu\nu}(x)-\eta_{\mu;\nu}-\eta_{\nu;\mu}+\eta^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$
$$g_{\mu\nu}(x')=g_{\mu\nu}(x+\eta)=g_{\mu\nu}(x)+\eta^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$
$$\bar{g}'_{\mu\nu}(x')=g_{\mu\nu}(x)+h_{\mu\nu}-\eta_{\mu;\nu}-\eta_{\nu;\mu}+\eta^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$
$$=g_{\mu\nu}(x')+h'_{\mu\nu}$$
由於我們要求總度規在經過座標變換後的背景度規依然是 Schwarzschild 度規，因此 $h'_{\mu\nu}=h_{\mu\nu}-\eta_{\mu;\nu}-\eta_{\nu;\mu}$。現在建構一個 axial 規範向量如下
$$\eta_{\mu}=\Lambda(t,r)\left[0,0,\overset{2}{V}_{lm}(\theta,\varphi)\right]=\Lambda(t,r)\left[0,0,-\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{lm}(\theta,\varphi),\sin\theta\frac{\partial}{\partial\theta}Y_{lm}(\theta,\varphi)\right]$$
$$\eta_{\mu;\nu}+\eta_{\nu;\mu}=\frac{\partial\eta_{\mu}}{\partial x^{\nu}}+\frac{\partial\eta_{\nu}}{\partial x^{\mu}}-2\eta_{\lambda}\Gamma^{\lambda}{}_{\mu\nu}$$
$$=\frac{\partial\eta_{\mu}}{\partial x^{\nu}}+\frac{\partial\eta_{\nu}}{\partial x^{\mu}}-\eta^{\alpha}\delta^{k}_{\alpha}\left(\frac{\partial g_{\mu k}}{\partial x^{\nu}}+\frac{\partial g_{\nu k}}{\partial x^{\mu}}-\frac{\partial g_{\mu\nu}}{\partial x^{k}}\right)$$
$$=\frac{\partial\eta_{\mu}}{\partial x^{\nu}}+\frac{\partial\eta_{\nu}}{\partial x^{\mu}}-\eta^{\alpha}\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}}-\eta^{\alpha}\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}}+\eta^{\alpha}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$
$$=\frac{\partial\eta_{\mu}}{\partial x^{\nu}}+\frac{\partial\eta_{\nu}}{\partial x^{\mu}}-\eta_{\beta}g^{\alpha\beta}\frac{\partial g_{\mu\alpha}}{\partial x^{\nu}}-\eta_{\beta}g^{\alpha\beta}\frac{\partial g_{\nu\alpha}}{\partial x^{\mu}}+\eta_{\beta}g^{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x^{\alpha}}$$

$$\therefore h'_{\mu\nu} = h_{\mu\nu} - \frac{\partial \eta_\mu}{\partial x^\nu} - \frac{\partial \eta_\nu}{\partial x^\mu} + \eta_\beta g^{\alpha\beta}\frac{\partial g_{\mu\alpha}}{\partial x^\nu} + \eta_\beta g^{\alpha\beta}\frac{\partial g_{\nu\alpha}}{\partial x^\mu} - \eta_\beta g^{\alpha\beta}\frac{\partial g_{\mu\nu}}{\partial x^\alpha}$$

$$\begin{aligned} h'_{02} &= h_{02} - \frac{\partial \eta_0}{\partial x^2} - \frac{\partial \eta_2}{\partial x^0} + \eta_0 g^{00}\frac{\partial g_{00}}{\partial x^2} + \eta_2 g^{22}\frac{\partial g_{22}}{\partial x^0} - \eta_\beta g^{\alpha\beta}\frac{\partial g_{02}}{\partial x^\alpha} \\ &= -\left(h_0 - \frac{\partial \Lambda}{\partial t}\right)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} \\ &= -h'_0 \frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} \end{aligned}$$

$h_0$ 的改變

$$\delta h_0 \equiv h'_0 - h_0 = -\frac{\partial}{\partial t}\Lambda(t,r)$$

同理可得 $h_1$ 與 $h_2$ 的改變

$$\delta h_1 \equiv h'_1 - h_1 = -\frac{\partial}{\partial r}\Lambda(t,r) + 2\frac{\Lambda(t,r)}{r}$$

$$\delta h_2 \equiv h'_2 - h_2 = -2\Lambda(t,r)$$

定義 $k_1$

$$k_1 \equiv h_1 - \frac{1}{2}\left(\frac{\partial h_2}{\partial r} - 2\frac{h_2}{r}\right)$$

在座標變換下

$$\begin{aligned} \delta k_1 &\equiv k'_1 - k_1 \\ &= \left(h'_1 - h_1\right) - \frac{1}{2}\left[\frac{\partial}{\partial r}\left(h'_2 - h_2\right) - \frac{2}{r}\left(h'_2 - h_2\right)\right] \\ &= \left(\delta h_1\right) - \frac{1}{2}\left[\frac{\partial}{\partial r}\left(\delta h_2\right) - \frac{2}{r}\left(\delta h_2\right)\right] \\ &= 0 \end{aligned}$$

因此 $k_1$ 是一個規範不變函數。Regge 與 Wheeler 利用規範自由度以消去角度 $(\theta,\varphi)$ 的最高階導數的貢獻。對於 axial 微擾，這指的是選擇一個規範使得 $h_2(t,r)$ 為 0，即使 $h_2(t,r)$ 原來並不為 0，我們還是可以令 $\Lambda(t,r) = \frac{1}{2}h_2(t,r)$ 以達成此目的。因此在 Regge-Wheeler 規範下

$$k_1(t,r) = \left(h_1\right)_{RW}(t,r)$$

如此一來 axial 微擾度規 $h_{\mu\nu}$ 可簡化成

$$\left(h_{\mu\nu}\right) = \begin{pmatrix} 0 & 0 & -h_0(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} & h_0(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial \theta} \\ 0 & 0 & -h_1(t,r)\frac{1}{\sin\theta}\frac{\partial Y_{lm}}{\partial \varphi} & h_1(t,r)\sin\theta\frac{\partial Y_{lm}}{\partial \theta} \\ * & * & 0 & 0 \\ * & * & 0 & 0 \end{pmatrix}$$

## 2.3 Axial 重力微擾方程式

由(2.8)式可知

$$\delta R_{23} = 0$$
$$\delta R_{13} = 0$$
$$\delta R_{03} = 0$$

由(2.7)式及(2.10)式可得

$$\delta R_{23} = \delta\Gamma^{\alpha}{}_{2\alpha,3} - \delta\Gamma^{\alpha}{}_{23,\alpha} + \delta\Gamma^{\beta}{}_{2\alpha}\Gamma^{\alpha}{}_{3\beta} + \Gamma^{\beta}{}_{2\alpha}\delta\Gamma^{\alpha}{}_{3\beta} - \delta\Gamma^{\beta}{}_{23}\Gamma^{\alpha}{}_{\alpha\beta} - \Gamma^{\beta}{}_{23}\delta\Gamma^{\alpha}{}_{\alpha\beta}$$

$$\delta\Gamma^{\alpha}{}_{2\alpha,3} = \frac{\partial}{\partial x^3}\left(\delta\Gamma^{\alpha}{}_{2\alpha}\right) = \frac{\partial}{\partial x^3}\left(\delta\Gamma^{0}{}_{20} + \delta\Gamma^{1}{}_{21} + \delta\Gamma^{2}{}_{22} + \delta\Gamma^{3}{}_{23}\right) = 0$$

$$\begin{aligned}\delta\Gamma^{\alpha}{}_{23,\alpha} &= \frac{\partial}{\partial x^\alpha}\left(\delta\Gamma^{\alpha}{}_{23}\right) = \frac{\partial}{\partial x^0}\left(\delta\Gamma^{0}{}_{23}\right) + \frac{\partial}{\partial x^1}\left(\delta\Gamma^{1}{}_{23}\right) + \frac{\partial}{\partial x^2}\left(\delta\Gamma^{2}{}_{23}\right) + \frac{\partial}{\partial x^3}\left(\delta\Gamma^{3}{}_{23}\right) \\ &= \frac{1}{2}\left[\left(\frac{2M}{r^2}\right)h_1 + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial h_1}{\partial r}\right) - \frac{1}{(1-2M/r)}\left(\frac{\partial h_0}{\partial t}\right)\right]\sin\theta W_{lm}(\theta,\varphi)\end{aligned}$$

$$\delta\Gamma^{\beta}{}_{2\alpha}\Gamma^{\alpha}{}_{3\beta} + \Gamma^{\beta}{}_{2\alpha}\delta\Gamma^{\alpha}{}_{3\beta} - \delta\Gamma^{\beta}{}_{23}\Gamma^{\alpha}{}_{\alpha\beta} - \Gamma^{\beta}{}_{23}\delta\Gamma^{\alpha}{}_{\alpha\beta} = 0$$

$$\begin{aligned}\delta R_{23} &= -\frac{1}{2}\left[\left(\frac{2M}{r^2}\right)h_1 + \left(1 - \frac{2M}{r}\right)\left(\frac{\partial h_1}{\partial r}\right) - \frac{1}{(1-2M/r)}\left(\frac{\partial h_0}{\partial t}\right)\right]\sin\theta W_{lm}(\theta,\varphi) \\ &= \frac{1}{2}\left\{\frac{1}{(1-2M/r)}\left(\frac{\partial h_0}{\partial t}\right) - \frac{\partial}{\partial r}\left[\left(1 - \frac{2M}{r}\right)h_1\right]\right\}\sin\theta W_{lm}(\theta,\varphi) \\ &= 0\end{aligned}$$

$$\therefore \frac{1}{(1-2M/r)}\left(\frac{\partial h_0}{\partial t}\right) - \frac{\partial}{\partial r}\left[\left(1 - \frac{2M}{r}\right)h_1\right] = 0 \qquad (2.12)$$

$$\delta R_{13} = \delta\Gamma^{\alpha}{}_{1\alpha,3} - \delta\Gamma^{\alpha}{}_{13,\alpha} + \delta\Gamma^{\beta}{}_{1\alpha}\Gamma^{\alpha}{}_{3\beta} + \Gamma^{\beta}{}_{1\alpha}\delta\Gamma^{\alpha}{}_{3\beta} - \delta\Gamma^{\beta}{}_{13}\Gamma^{\alpha}{}_{\alpha\beta} - \Gamma^{\beta}{}_{13}\delta\Gamma^{\alpha}{}_{\alpha\beta}$$

$$\delta\Gamma^{\alpha}{}_{1\alpha,3} = \frac{\partial}{\partial x^3}\left(\delta\Gamma^{\alpha}{}_{1\alpha}\right) = \frac{\partial}{\partial x^3}\left(\delta\Gamma^{0}{}_{10} + \delta\Gamma^{1}{}_{11} + \delta\Gamma^{2}{}_{12} + \delta\Gamma^{3}{}_{13}\right) = 0$$

$$\begin{aligned}\delta\Gamma^{\alpha}{}_{13,\alpha} &= \frac{\partial}{\partial x^\alpha}\left(\delta\Gamma^{\alpha}{}_{13}\right) = \frac{\partial}{\partial x^0}\left(\delta\Gamma^{0}{}_{13}\right) + \frac{\partial}{\partial x^1}\left(\delta\Gamma^{1}{}_{13}\right) + \frac{\partial}{\partial x^2}\left(\delta\Gamma^{2}{}_{13}\right) + \frac{\partial}{\partial x^3}\left(\delta\Gamma^{3}{}_{13}\right) \\ &= \frac{1}{2}\frac{1}{(1-2M/r)}\left[\frac{\partial^2 h_1}{\partial t^2} - \frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0 + \frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ &\quad + \left[-\frac{2M}{r^2}\frac{h_1}{r} + \left(1 - \frac{2M}{r}\right)\frac{h_1}{r^2} - \left(1 - \frac{2M}{r}\right)\frac{1}{r}\left(\frac{\partial h_1}{\partial r}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta} \\ &\quad + \frac{1}{2}\frac{h_1}{r^2}l(l+1)\left(\cos\theta Y_{lm} + \sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right)\end{aligned}$$

$$\delta\Gamma^{\beta}{}_{1\alpha}\Gamma^{\alpha}{}_{3\beta}+\Gamma^{\beta}{}_{1\alpha}\delta\Gamma^{\alpha}{}_{3\beta}-\delta\Gamma^{\beta}{}_{13}\Gamma^{\alpha}{}_{\alpha\beta}-\Gamma^{\beta}{}_{13}\delta\Gamma^{\alpha}{}_{\alpha\beta}$$
$$=\left[-\left(1-\frac{2M}{r}\right)\frac{1}{r}\left(\frac{\partial h_1}{\partial r}\right)+\left(1-\frac{2M}{r}\right)\frac{2}{r^2}h_1\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}+\frac{1}{2}\frac{h_1}{r^2}l(l+1)\cos\theta Y_{lm}$$

$$\begin{aligned}
\delta R_{13}&=-\frac{1}{2}\frac{1}{(1-2M/r)}\left[\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0+\frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\
&+\left[\frac{2M}{r^2}\frac{h_1}{r}-\left(1-\frac{2M}{r}\right)\frac{h_1}{r^2}+\left(1-\frac{2M}{r}\right)\frac{1}{r}\left(\frac{\partial h_1}{\partial r}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\
&-\frac{1}{2}\frac{h_1}{r^2}l(l+1)\left(\cos\theta Y_{lm}+\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right)\\
&+\left[-\left(1-\frac{2M}{r}\right)\frac{1}{r}\left(\frac{\partial h_1}{\partial r}\right)+\left(1-\frac{2M}{r}\right)\frac{2}{r^2}h_1\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}+\frac{1}{2}\frac{h_1}{r^2}l(l+1)\cos\theta Y_{lm}\\
&=-\frac{1}{2}\left\{\frac{1}{(1-2M/r)}\left[\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0+\frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)\right]+\frac{1}{r^2}\left[l(l+1)-2\right]h_1\right\}\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\
&=0
\end{aligned}$$

$$\therefore\frac{1}{(1-2M/r)}\left[\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0+\frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)\right]+\frac{1}{r^2}\left[l(l+1)-2\right]h_1=0 \qquad (2.13)$$

$$\delta R_{03}=\delta\Gamma^{\alpha}{}_{0\alpha,3}-\delta\Gamma^{\alpha}{}_{03,\alpha}+\delta\Gamma^{\beta}{}_{0\alpha}\Gamma^{\alpha}{}_{3\beta}+\Gamma^{\beta}{}_{0\alpha}\delta\Gamma^{\alpha}{}_{3\beta}-\delta\Gamma^{\beta}{}_{03}\Gamma^{\alpha}{}_{\alpha\beta}-\Gamma^{\beta}{}_{03}\delta\Gamma^{\alpha}{}_{\alpha\beta}$$

$$\delta\Gamma^{\alpha}{}_{0\alpha,3}=\frac{\partial}{\partial x^3}\left(\delta\Gamma^{\alpha}{}_{0\alpha}\right)=\frac{\partial}{\partial x^3}\left(\delta\Gamma^{0}{}_{00}+\delta\Gamma^{1}{}_{01}+\delta\Gamma^{2}{}_{02}+\delta\Gamma^{3}{}_{03}\right)=0$$
$$\begin{aligned}
\delta\Gamma^{\alpha}{}_{03,\alpha}&=\frac{\partial}{\partial x^\alpha}\left(\delta\Gamma^{\alpha}{}_{03}\right)=\frac{\partial}{\partial x^0}\left(\delta\Gamma^{0}{}_{03}\right)+\frac{\partial}{\partial x^1}\left(\delta\Gamma^{1}{}_{03}\right)+\frac{\partial}{\partial x^2}\left(\delta\Gamma^{2}{}_{03}\right)+\frac{\partial}{\partial x^3}\left(\delta\Gamma^{3}{}_{03}\right)\\
&=\left[\frac{1}{2}\frac{2M}{r^2}\left(\frac{\partial h_1}{\partial t}-\frac{\partial h_0}{\partial r}\right)+\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_1-\frac{\partial^2 h_0}{\partial r^2}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\\
&+\frac{1}{2}\frac{h_0}{r^2}l(l+1)\left(\cos\theta Y_{lm}+\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right)
\end{aligned}$$
$$\delta\Gamma^{\beta}{}_{0\alpha}\Gamma^{\alpha}{}_{3\beta}+\Gamma^{\beta}{}_{0\alpha}\delta\Gamma^{\alpha}{}_{3\beta}-\delta\Gamma^{\beta}{}_{03}\Gamma^{\alpha}{}_{\alpha\beta}-\Gamma^{\beta}{}_{03}\delta\Gamma^{\alpha}{}_{\alpha\beta}$$
$$=\left[-\frac{1}{r}\left(1-\frac{2M}{r}\right)\left(\frac{\partial h_1}{\partial t}\right)+\frac{2M}{r^2}\frac{h_0}{r}+\frac{1}{2}\frac{2M}{r^2}\left(\frac{\partial h_1}{\partial t}-\frac{\partial h_0}{\partial r}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}$$
$$+\frac{1}{2}\frac{h_0}{r^2}l(l+1)\cos\theta Y_{lm}$$

$$\delta R_{03}=\left[-\frac{1}{2}\frac{2M}{r^2}\left(\frac{\partial h_1}{\partial t}-\frac{\partial h_0}{\partial r}\right)-\frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_1-\frac{\partial^2 h_0}{\partial r^2}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}$$

$$-\frac{1}{2}\frac{h_0}{r^2}l(l+1)\left(\cos\theta Y_{lm}+\sin\theta\frac{\partial Y_{lm}}{\partial\theta}\right)$$

$$+\left[-\frac{1}{r}\left(1-\frac{2M}{r}\right)\left(\frac{\partial h_1}{\partial t}\right)+\frac{2M}{r^2}\frac{h_0}{r}+\frac{1}{2}\frac{2M}{r^2}\left(\frac{\partial h_1}{\partial t}-\frac{\partial h_0}{\partial r}\right)\right]\sin\theta\frac{\partial Y_{lm}}{\partial\theta}$$

$$+\frac{1}{2}\frac{h_0}{r^2}l(l+1)\cos\theta Y_{lm}$$

$$=\left\{\frac{1}{2}\left(1-\frac{2M}{r}\right)\left[\frac{\partial^2 h_0}{\partial r^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_1-\frac{2}{r}\left(\frac{\partial h_1}{\partial t}\right)\right]+\frac{1}{r^2}\left[\frac{2M}{r}-\frac{1}{2}l(l+1)\right]h_0\right\}\sin\theta\frac{\partial Y_{lm}}{\partial\theta}$$

$$=\left\{\begin{aligned}&\frac{1}{2}\left(1-\frac{2M}{r}\right)\left[\frac{\partial^2 h_0}{\partial r^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_1-\frac{2}{r}\left(\frac{\partial h_1}{\partial t}\right)\right]\\&+\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(1-\frac{2M}{r}\right)-\frac{1}{2}l(l+1)\right]h_0\end{aligned}\right\}\sin\theta\frac{\partial Y_{lm}}{\partial\theta}$$

$$=0$$

$$\therefore\frac{1}{2}\left(1-\frac{2M}{r}\right)\left[\frac{\partial^2 h_0}{\partial r^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_1-\frac{2}{r}\left(\frac{\partial h_1}{\partial t}\right)\right]+\frac{1}{r^2}\left[r\frac{\partial}{\partial r}\left(1-\frac{2M}{r}\right)-\frac{1}{2}l(l+1)\right]h_0=0\quad(2.14)$$

現在我們有(2.12)、(2.13)與(2.14)三個微分方程式可用來解兩個未知函數 $h_0(t,r)$ 與 $h_1(t,r)$，但是方程式(2.14)可以從方程式(2.12)及(2.13)導出，所以實際上只有(2.12)與(2.13)兩個互相獨立的方程式用來解兩個未知函數 $h_0(t,r)$ 與 $h_1(t,r)$。令

$$B(r)=1-\frac{2M}{r}\qquad(2.15)$$

代入(2.12)式

$$\begin{aligned}&\frac{1}{B(r)}\left(\frac{\partial h_0}{\partial t}\right)-\frac{\partial}{\partial r}\left[B(r)h_1\right]=0\\&\Rightarrow\frac{\partial h_0}{\partial t}=B(r)h_1\left[\frac{\partial}{\partial r}B(r)\right]+B^2(r)\left(\frac{\partial h_1}{\partial r}\right)\end{aligned}\qquad(2.16)$$

將(2.15)式及(2.16)式代入(2.13)式

$$\frac{1}{B(r)}\left[\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0+\frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)\right]+\frac{1}{r^2}\left[l(l+1)-2\right]h_1=0$$

$$\Rightarrow \frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial t}\frac{\partial}{\partial r}h_0+\frac{2}{r}\left(\frac{\partial h_0}{\partial t}\right)+\frac{B(r)}{r^2}\left[l(l+1)-2\right]h_1=0$$

$$\Rightarrow \left[\begin{array}{l}\frac{\partial^2 h_1}{\partial t^2}-\frac{\partial}{\partial r}\left\{B(r)h_1\left[\frac{\partial}{\partial r}B(r)\right]+B^2(r)\left(\frac{\partial h_1}{\partial r}\right)\right\}\\ +\frac{2}{r}\left\{B(r)h_1\left[\frac{\partial}{\partial r}B(r)\right]+B^2(r)\left(\frac{\partial h_1}{\partial r}\right)\right\}+\frac{B(r)}{r^2}\left[l(l+1)-2\right]h_1\end{array}\right]=0$$

$$\Rightarrow \left[\begin{array}{l}\frac{\partial^2 h_1}{\partial t^2}-\left[\frac{\partial}{\partial r}B(r)\right]^2 h_1-3B(r)\left[\frac{\partial}{\partial r}B(r)\right]\left(\frac{\partial h_1}{\partial r}\right)\\ -B(r)\left[\frac{\partial^2}{\partial r^2}B(r)\right]h_1-B^2(r)\left(\frac{\partial^2 h_1}{\partial r^2}\right)\\ +\frac{2B(r)}{r}\left[\frac{\partial}{\partial r}B(r)\right]h_1+\frac{2B^2(r)}{r}\left(\frac{\partial h_1}{\partial r}\right)+\frac{B(r)}{r^2}\left[l(l+1)-2\right]h_1\end{array}\right]=0$$

令 $h_1(t,r)=\frac{1}{B(r)}rQ_l(t,r)$ ，上式變成

$$\frac{\partial^2 Q_l}{\partial t^2}-\frac{B(r)}{r}\left\{\frac{\partial}{\partial r}\left\{B(r)\left[\frac{\partial}{\partial r}(rQ_l)\right]\right\}\right\}+\frac{2B^2(r)}{r^2}\left[\frac{\partial}{\partial r}(rQ_l)\right]+\frac{B(r)}{r^2}\left[l(l+1)-2\right]Q_l=0$$

令 $Q_l(t,r)=b(r)q_l(t,x(r))$ ，上式變成

$$\begin{aligned}&\left(1-\frac{2M}{r}\right)\left[\left(\frac{l(l+1)}{r^2}-\frac{6M}{r^3}\right)b(r)-\frac{2M}{r^2}\left(\frac{db}{dr}\right)-\left(1-\frac{2M}{r}\right)\left(\frac{d^2b}{dr^2}\right)\right]q_l(t,x(r))\\ &-\left(1-\frac{2M}{r}\right)\left\{\left[\frac{2M}{r^2}\left(\frac{dx}{dr}\right)+\left(1-\frac{2M}{r}\right)\left(\frac{d^2x}{dr^2}\right)\right]b(r)+2\left(1-\frac{2M}{r}\right)\left(\frac{dx}{dr}\right)\left(\frac{db}{dr}\right)\right\}\frac{\partial q_l}{\partial x}\\ &-\left[\left(1-\frac{2M}{r}\right)^2\left(\frac{dx}{dr}\right)^2 b(r)\right]\frac{\partial^2 q_l}{\partial x^2}+b(r)\frac{\partial^2 q_l}{\partial t^2}=0\end{aligned} \qquad (2.17)$$

為了將上式化成 Schrödinger 方程式的形式，令

$$\left(1-\frac{2M}{r}\right)^2\left(\frac{dx}{dr}\right)^2 b(r)=b(r) \qquad (2.18)$$

$$\left[\frac{2M}{r^2}\left(\frac{dx}{dr}\right)+\left(1-\frac{2M}{r}\right)\left(\frac{d^2x}{dr^2}\right)\right]b(r)+2\left(1-\frac{2M}{r}\right)\left(\frac{dx}{dr}\right)\left(\frac{db}{dr}\right)=0 \qquad (2.19)$$

從(2.18)式得 $\frac{dx}{dr}=\pm\frac{1}{(1-2M/r)}$ ，若取 $\frac{dx}{dr}=\frac{1}{(1-2M/r)}$ ，可得

$$x = r + 2M\ln\left(\frac{r}{2M} - 1\right)$$

從(2.19)式可得$b(r) = const.$，令$b(r) = 1$，故$Q_l(t, r(x)) = q_l(t, x)$，(2.17)式變成

$$\frac{\partial^2}{\partial t^2}Q_l(t,x) - \frac{\partial^2}{\partial x^2}Q_l(t,x) + V_{RW}(x)Q_l(t,x) = 0 \qquad (2.20)$$

其中

$$V_{RW}(x) = \left(1 - \frac{2M}{r(x)}\right)\left[\frac{l(l+1)}{r(x)^2} - \frac{6M}{r(x)^3}\right]$$

$V_{RW}$稱為 Regge-Wheeler 位能，(2.20)式是描述 axial 微擾的一維波動方程式，稱為 Regge-Wheeler 方程式。

## 2.4 在 Laplace 轉換下所得出的 quasinormal modes

Regge-Wheeler 方程式的解$Q_l(t,x)$的 Laplace 轉換如下

$$\hat{f}(s,x) = \int_0^\infty e^{-st}Q_l(t,x)dt$$

$\hat{f}(s,x)$是一個對於$\mathrm{Re}(s) > 0$的$s$的解析函數。將 Regge-Wheeler 方程式等號兩邊同乘以$e^{-st}$後再對$t$做積分

$$\int_0^\infty e^{-st}\frac{\partial^2}{\partial t^2}Q_l(t,x)dt - \int_0^\infty e^{-st}\frac{\partial^2}{\partial x^2}Q_l(t,x)dt + \int_0^\infty e^{-st}V_{RW}(x)Q_l(t,x)dt = 0$$

其中

$$\int_0^\infty e^{-st}\frac{\partial^2}{\partial t^2}Q_l(t,x)dt = -sQ_l\Big|_{t=0} - \frac{\partial Q_l}{\partial t}\Big|_{t=0} + s^2\int_0^\infty e^{-st}Q_l(t,x)dt$$

代入 Regge-Wheeler 方程式

$$\begin{bmatrix} -sQ_l\Big|_{t=0} - \dfrac{\partial Q_l}{\partial t}\Big|_{t=0} + s^2\displaystyle\int_0^\infty e^{-st}Q_l(t,x)dt \\ -\dfrac{\partial^2}{\partial x^2}\displaystyle\int_0^\infty e^{-st}Q_l(t,x)dt + V_{RW}(x)\displaystyle\int_0^\infty e^{-st}Q_l(t,x)dt \end{bmatrix} = 0$$

可得$Q_l(t,x)$的 Laplace 轉換$\hat{f}(s,x)$滿足一個微分方程式

$$\frac{\partial^2}{\partial x^2}\hat{f}(s,x) + \left(-s^2 - V_{RW}(x)\right)\hat{f}(s,x) = -sQ_l\Big|_{t=0} - \frac{\partial Q_l}{\partial t}\Big|_{t=0} \equiv I(s,x) \qquad (2.21)$$

(2.21)式是一個 inhomogeneous 微分方程式，其對應的 Green 函數$G(s,x,x')$滿足下列微分方程式

$$\frac{\partial^2}{\partial x^2}G(s,x,x') + \left(-s^2 - V_{RW}(x)\right)G(s,x,x') = \delta(x - x') \qquad (2.22)$$

從(2.21)式及(2.22)式可得

$$\hat{f}(s,x)=\int_{-\infty}^{\infty}G(s,x,x')I(s,x')dx'$$

若有一個 homogeneous 微分方程式如下

$$\frac{\partial^2}{\partial x^2}f(s,x)+\left(-s^2-V_{RW}(x)\right)f(s,x)=0 \qquad (2.23)$$

假設滿足(2.23)式的兩個線性獨立的解為 $f_+$ 與 $f_-$，則我們可藉由這兩個線性獨立的解來建構出 Green 函數 $G(s,x,x')$。若 $x>x'$，則

$$\frac{\partial^2}{\partial x^2}G(s,x,x')+\left(-s^2-V_{RW}(x)\right)G(s,x,x')=0$$

上式的解為滿足 $x>x'$ 的邊界條件，必須是

$$G(s,x,x')=c_1f_+(s,x)$$

若 $x<x'$，則

$$\frac{\partial^2}{\partial x^2}G(s,x,x')+\left(-s^2-V_{RW}(x)\right)G(s,x,x')=0$$

上式的解為滿足 $x<x'$ 的邊界條件，必須是

$$G(s,x,x')=c_2f_-(s,x)$$

在 $x=x'$ 處

$$c_1f_+(s,x')=c_2f_-(s,x') \qquad (2.24)$$

$$\int_{x'-\varepsilon}^{x'+\varepsilon}\frac{\partial^2}{\partial x^2}G(s,x,x')dx+\int_{x'-\varepsilon}^{x'+\varepsilon}\left(-s^2-V_{RW}(x)\right)G(s,x,x')dx=\int_{x'-\varepsilon}^{x'+\varepsilon}\delta(x-x')dx \qquad (2.25)$$

其中 $\varepsilon\to 0$，可得

$$\int_{x'-\varepsilon}^{x'+\varepsilon}\left(-s^2-V_{RW}(x)\right)G(s,x,x')dx\cong\left(-s^2-V_{RW}(x')\right)G(s,x')2\varepsilon\cong 0$$

因此(2.25)式變成

$$\frac{\partial}{\partial x}G(s,x,x')\bigg|_{x=x'+\varepsilon}-\frac{\partial}{\partial x}G(s,x,x')\bigg|_{x=x'-\varepsilon}=1$$

$$\Rightarrow c_1\frac{\partial}{\partial x}f_+(s,x)\bigg|_{x=x'}-c_2\frac{\partial}{\partial x}f_-(s,x)\bigg|_{x=x'}=1$$

將上式簡寫為

$$c_1f_+'(s,x')-c_2f_-'(s,x')=1 \qquad (2.26)$$

由(2.24)式及(2.26)式聯立解得

$$c_1=\frac{f_-(s,x')}{W(s)}$$

$$c_2=\frac{f_+(s,x')}{W(s)}$$

其中

$$W(s) = f_-(s,x')f'_+(s,x') - f'_-(s,x')f_+(s,x')$$
$$= f_-(s,x)f'_+(s,x) - f'_-(s,x)f_+(s,x)$$

在 $x > x'$ 處

$$G(s,x,x') = \frac{f_-(s,x')f_+(s,x)}{W(s)} \qquad (2.27)$$

在 $x < x'$ 處

$$G(s,x,x') = \frac{f_+(s,x')f_-(s,x)}{W(s)} \qquad (2.28)$$

(2.27)式及(2.28)式可合併為一式

$$G(s,x,x') = \frac{f_-(s,x_<)f_+(s,x_>)}{W(s)}$$

其中 $x_< \equiv \min(x',x)$ ， $x_> \equiv \max(x',x)$ ， $W(s) = f_-(s,x)f'_+(s,x) - f'_-(s,x)f_+(s,x)$ 是 Wronskian，與變數 $x$ 無關，可確認如下

$$\begin{cases} \dfrac{\partial^2}{\partial x^2} f_+(s,x) + \left(-s^2 - V_{RW}(x)\right) f_+(s,x) = 0 \\ \dfrac{\partial^2}{\partial x^2} f_-(s,x) + \left(-s^2 - V_{RW}(x)\right) f_-(s,x) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} f_-(s,x)\left[\dfrac{\partial^2}{\partial x^2} f_+(s,x) + \left(-s^2 - V_{RW}(x)\right) f_+(s,x)\right] = 0 \\ f_+(s,x)\left[\dfrac{\partial^2}{\partial x^2} f_-(s,x) + \left(-s^2 - V_{RW}(x)\right) f_-(s,x)\right] = 0 \end{cases}$$

兩式相減可得

$$f_-(s,x)\frac{\partial^2}{\partial x^2} f_+(s,x) - f_+(s,x)\frac{\partial^2}{\partial x^2} f_-(s,x) = 0$$
$$\Rightarrow \frac{\partial}{\partial x}\left[f_-(s,x)\frac{\partial}{\partial x} f_+(s,x) - f_+(s,x)\frac{\partial}{\partial x} f_-(s,x)\right] = 0$$

故得 $W(s) = f_-(s,x)\dfrac{\partial}{\partial x} f_+(s,x) - f_+(s,x)\dfrac{\partial}{\partial x} f_-(s,x)$ 是一與變數 $x$ 無關的函數。可從 Laplace 轉換得到 $\hat{f}(s,x)$ 的 Inverse Laplace 轉換 $Q_l(t,x)$ 如下

$$Q_l(t,x) = \frac{1}{2\pi i}\int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{st}\hat{f}(s,x)ds$$

為了討論 quasinormal modes，我們假設沿著 contour 的半圓的積分值為零，且在 contour 內沒有本性奇點（essential singularity），且 $f_+$ 與 $f_-$ 是 $s$ 的解析函數，且 Green 函數的極點（poles）源於 Wronskian 的根，並且這些極點是單純極點（simple poles），則

$$
\begin{aligned}
Q_l(t,x) &= \frac{1}{2\pi i}\oint e^{st}\hat{f}(s,x)ds \\
&= \frac{1}{2\pi i}\oint e^{st}\int_{-\infty}^{\infty} G(s,x,x')I(s,x')dx'ds \\
&= \frac{1}{2\pi i}\oint\left(e^{st}\frac{1}{W(s)}\int_{-\infty}^{\infty} f_-(s,x_<)f_+(s,x_>)I(s,x')dx'\right)ds \\
&= \sum_q \mathrm{Res}\left(e^{st}\frac{1}{W(s)}\int_{-\infty}^{\infty} f_-(s,x_<)f_+(s,x_>)I(s,x')dx', s_q\right) \\
&= \sum_q e^{s_q t}\frac{1}{\left(dW(s)/ds\right)\big|_{s=s_q}}\int_{-\infty}^{\infty} f_-(s_q,x_<)f_+(s_q,x_>)I(s_q,x')dx'
\end{aligned}
\tag{2.29}
$$

若 initial data 有 compact support，且 $x > x'$ ，則

$$
Q_l(t,x) = \sum_q c_q u_q(t,x)
$$

其中

$$
c_q = \frac{1}{\left(dW(s)/ds\right)\big|_{s=s_q}}\int_{x_l}^{x_r} f_-(s_q,x')I(s_q,x')dx'
$$

$$
u_q(t,x) = e^{s_q t} f_+(s_q,x)
$$

$x_l$ 與 $x_r$ 表示 initial data 的 compact support 的左邊界與右邊界。在這些極點上

$$
\begin{aligned}
& W(s=s_q) = 0 = f_-(s_q,x)f_+'(s_q,x) - f_-'(s_q,x)f_+(s_q,x) \\
& \Rightarrow f_-(s_q,x) = f_+(s_q,x)
\end{aligned}
\tag{2.30}
$$

當 $x \to \pm\infty$ 時，$V_{RW}(x) \to 0$ ，(2.23)式變成

$$
\frac{\partial^2}{\partial x^2} f(s,x) - s^2 f(s,x) = 0
$$

可解得 $f(s,x) \sim e^{\pm sx}$ ， $\mathrm{Re}(s) > 0$ ，若選擇束縛態的解，則

$$
\begin{aligned}
f_-(s,x) &\sim e^{sx} \qquad \text{as} \quad x \to -\infty \\
f_+(s,x) &\sim e^{-sx} \qquad \text{as} \quad x \to \infty
\end{aligned}
\tag{2.31}
$$

若取 $s = -i\omega$，(2.23)式成為 Regge-Wheeler 方程式經過 Fourier 轉換之後的微分方程式，由(2.30)式知當 $s = s_q$ 時 $f_- = f_+$ ，因此它同時滿足(2.31)式的 out-going 邊界條件。一般而言，選擇 out-going 解的邊界條件會使得本徵值 $\omega$ 為複數值，因此我們把這些模態稱為 quasinormal modes。

## 2.5 Axial 重力微擾的反射率與透射率

將(2.20)式以完整單位的形式寫出

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}Q_l(t,x)-\frac{\partial^2}{\partial x^2}Q_l(t,x)+V_{RW}(x)Q_l(t,x)=0 \qquad (2.32)$$

$Q_l^*\times(2.32)-Q_l\times(2.32)^*$，可得出

$$\frac{1}{c^2}\left(Q_l^*\frac{\partial^2}{\partial t^2}Q_l-Q_l\frac{\partial^2}{\partial t^2}Q_l^*\right)-\left(Q_l^*\frac{\partial^2}{\partial x^2}Q_l-Q_l\frac{\partial^2}{\partial x^2}Q_l^*\right)=0$$

$$\frac{\partial}{\partial t}\left[-\frac{1}{c^2}\left(Q_l^*\frac{\partial}{\partial t}Q_l-Q_l\frac{\partial}{\partial t}Q_l^*\right)\right]+\frac{\partial}{\partial x}\left(Q_l^*\frac{\partial}{\partial x}Q_l-Q_l\frac{\partial}{\partial x}Q_l^*\right)=0$$

上式與連續性方程式$\frac{\partial}{\partial t}\rho+\frac{\partial}{\partial x}j=0$比較可得機率流密度

$$j\propto Q_l^*\frac{\partial}{\partial x}Q_l-Q_l\frac{\partial}{\partial x}Q_l^* \qquad (2.33)$$

現在用 normal-mode 分析，假設 axial 微擾方程式的解的形式如下

$$Q_l(t,x)=\exp(-i\omega t)\phi(x)$$

其中$\omega=E/\hbar$，(2.32)式變成

$$\frac{d^2\phi}{dx^2}+\left[\left(\frac{\omega}{c}\right)^2-V_{RW}(x)\right]\phi(x)=0 \qquad (2.34)$$

其中

$$V_{RW}(x)=\left(1-\frac{2GM/c^2}{r(x)}\right)\left[\frac{l(l+1)}{r(x)^2}-\frac{6GM/c^2}{r(x)^3}\right]$$

令$y=\frac{r}{2GM/c^2}-1$，(2.34)式變成

$$y(y+1)\frac{d^2\phi}{dy^2}+\frac{d\phi}{dy}+\left[4\zeta^2\frac{(y+1)^3}{y}-l(l+1)+\frac{3}{y+1}\right]\phi(y)=0 \qquad (2.35)$$

其中$\zeta=\frac{GME}{\hbar c^3}$，$V_{RW}$變成

$$V_{RW}=\frac{1}{(2GM/c^2)^2}\left[\frac{l(l+1)y}{(y+1)^3}-\frac{3y}{(y+1)^4}\right]$$

$l=2$ ，$2GM/c^2=1$ 的位能圖

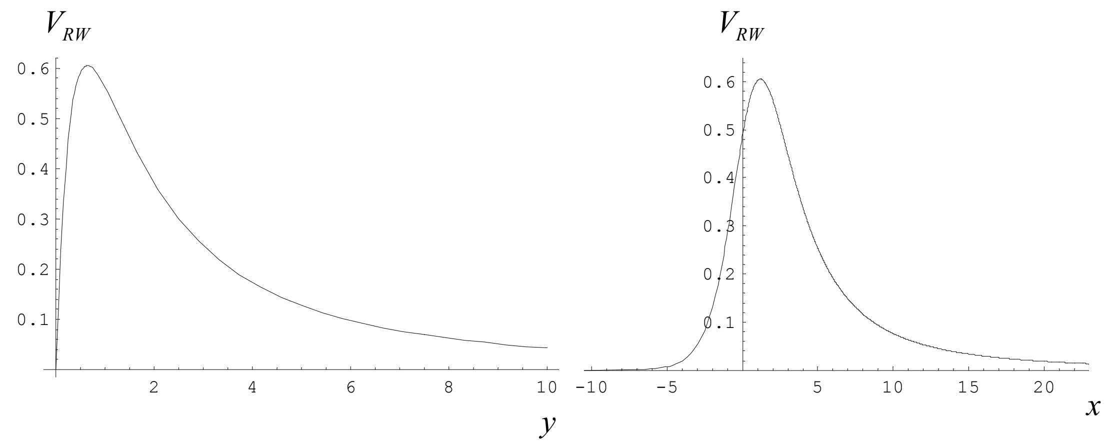


圖 2.1：左圖 horizon 位在 $y=0$ ，右圖 horizon 位在 $x=-\infty$ 。

已知 $x=\dfrac{2GM}{c^2}\left[y+1+\ln(y)\right]$，當 $x\to-\infty$（即 $y\to 0$ 或 $r\to 2GM/c^2$ ）時，與當 $x\to\infty$（即 $y\to\infty$ 或 $r\to\infty$ ）時，$V_{RW}\to 0$ ，(2.34)式可簡化成

$$\frac{d^2\phi}{dx^2}+\left(\frac{\omega}{c}\right)^2\phi=0$$

可解得

$$\phi=a_0\exp\{i2\zeta[y+\ln(y)]\}+b_0\exp\{-i2\zeta[y+\ln(y)]\} \qquad (2.36)$$

$$\phi=C_1\exp\{i2\zeta[y+\ln(y)]\}+C_2\exp\{-i2\zeta[y+\ln(y)]\} \qquad (2.37)$$

(2.36)式是當 $r\to 2GM/c^2$ 時的徑向波函數，(2.37)式是當 $r\to\infty$ 時的徑向波函數。

現在考慮一個不同於 out-going 解的物理情況，即 axial 重力微擾從距離黑洞無限遠處往黑洞視界正向入射，求其反射率與透射率。在此物理情況，邊界條件是假設在距離黑洞無限遠處的入射徑向波函數及反射徑向波函數為(2.37)式所示，透射徑向波函數為(2.36)式的第二項。

入射徑向波函數： $\phi_i(y)=C_2\exp\{-i2\zeta[y+\ln(y)]\}$

反射波徑向函數： $\phi_r(y)=C_1\exp\{i2\zeta[y+\ln(y)]\}$

透射徑向波函數： $\phi_t(y)=b_0\exp\{-i2\zeta[y+\ln(y)]\}$

由於波函數 $Q_l(t,x)=\exp(-i\omega t)\phi(x)$ ，可知入射波函數

$$Q_{l_i}=\exp(-i\omega t)C_2\exp\{-i2\zeta[y+\ln(y)]\}$$

由(2.33)式可知入射機率流密度

$$j_i \propto {Q_{l_i}}^* \frac{\partial}{\partial x} Q_{l_i} - Q_{l_i} \frac{\partial}{\partial x} {Q_{l_i}}^* = \frac{|C_2|^2(-i4\zeta)}{(2GM/c^2)}$$

已知反射波函數

$$Q_{l_r} = \exp(-i\omega t) C_1 \exp\{i2\zeta[y + \ln(y)]\}$$

由(2.33)式可知反射機率流密度

$$j_r \propto {Q_{l_r}}^* \frac{\partial}{\partial x} Q_{l_r} - Q_{l_r} \frac{\partial}{\partial x} {Q_{l_r}}^* = \frac{|C_1|^2(i4\zeta)}{(2GM/c^2)}$$

已知透射波函數

$$Q_{l_t} = \exp(-i\omega t) b_0 \exp\{-i2\zeta[y + \ln(y)]\}$$

由(2.33)式可知透射機率流密度

$$j_t \propto {Q_{l_t}}^* \frac{\partial}{\partial x} Q_{l_t} - Q_{l_t} \frac{\partial}{\partial x} {Q_{l_t}}^* = \frac{|b_0|^2(-i4\zeta)}{(2GM/c^2)}$$

反射率

$$R \equiv \left|\frac{j_r}{j_i}\right| = \left|\frac{\dfrac{|C_1|^2(i4\zeta)}{(2GM/c^2)}}{\dfrac{|C_2|^2(-i4\zeta)}{(2GM/c^2)}}\right| = \frac{|C_1|^2}{|C_2|^2}$$

透射率

$$T \equiv \left|\frac{j_t}{j_i}\right| = \left|\frac{\dfrac{|b_0|^2(-i4\zeta)}{(2GM/c^2)}}{\dfrac{|C_2|^2(i4\zeta)}{(2GM/c^2)}}\right| = \frac{|b_0|^2}{|C_2|^2}$$

我們可經由以下的過程確認機率守恆關係式 $R + T = 1$ 。$\phi^* \times (2.34) - \phi \times (2.34)^*$ ，可得出

$$\phi^* \frac{d^2\phi}{dx^2} - \phi \frac{d^2\phi^*}{dx^2} = 0$$

$$\Rightarrow \frac{d}{dx}\left(\phi^* \frac{d\phi}{dx} - \phi \frac{d\phi^*}{dx}\right) = 0$$

$$\Rightarrow \phi^* \frac{d\phi}{dx} - \phi \frac{d\phi^*}{dx} = const. \qquad \forall x$$

上式可再變成一與 $y$ 無關的式子

$$\frac{y}{y+1}\left(\phi^* \frac{d\phi}{dy} - \phi \frac{d\phi^*}{dy}\right) = const. \qquad \forall y \qquad (2.38)$$

當 $y \to \infty$ 時，徑向波函數

$$\phi(y) = C_1 \exp\{i2\zeta[y + \ln(y)]\} + C_2 \exp\{-i2\zeta[y + \ln(y)]\}$$

可得

$$\frac{y}{y+1}\left(\phi^* \frac{d\phi}{dy} - \phi \frac{d\phi^*}{dy}\right) = i4\zeta\left(|C_1|^2 - |C_2|^2\right)$$

當 $y \to 0$ 時，徑向波函數

$$\phi(y) = b_0 \exp\{-i2\zeta[y + \ln(y)]\}$$

可得

$$\frac{y}{y+1}\left(\phi^* \frac{d\phi}{dy} - \phi \frac{d\phi^*}{dy}\right) = -i4\zeta|b_0|^2$$

由(2.38)式可知

$$i4\zeta\left(|C_1|^2 - |C_2|^2\right) = -i4\zeta|b_0|^2$$

$$\Rightarrow \frac{|C_1|^2}{|C_2|^2} + \frac{|b_0|^2}{|C_2|^2} = 1$$

故得機率守恆關係式 $R + T = 1$ 。

下圖是利用數值計算方法所求得的$R-\zeta$及$T-\zeta$關係圖

$l=2$

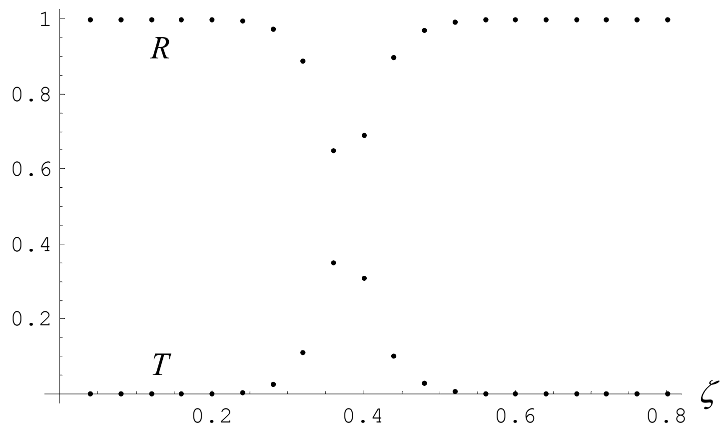


圖 2.2：反射率$R$隨$\zeta$的增加而下降，透射率$T$隨$\zeta$的增加而上升。當$\zeta \cong 0.4$時，$R=T=0.5$。

$l=3$

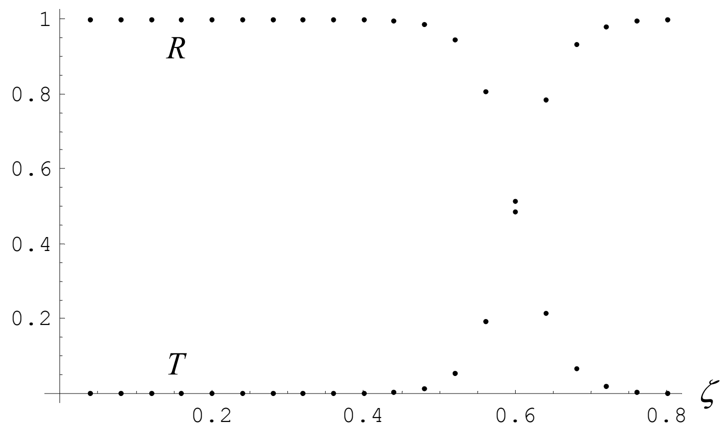


圖 2.3：反射率$R$隨$\zeta$的增加而下降，透射率$T$隨$\zeta$的增加而上升。當$\zeta \cong 0.6$時，$R=T=0.5$。