國立臺灣師範大學數學系博士班博士論文

指導教授： 陳界山 博士（指導教授）

施茂祥 博士（共同指導教授）

Generalization of Shih-Dong's combinational fixed point theorem to finite distributive lattices

研究生：吳樹恆

中華民國 104 年 7 月

Generalization of Shih-Dong's combinational fixed point theorem to finite distributive lattices

Shu-Han Wu

Abstract

Shih-Dong's combinational fixed point theorem asserts that if a map from the n-dimensional hypercube into itself satisfies that all the Boolean eigenvalues of the Boolean Jacobian matrix are zero for each element in the hypercube, then it has a unique fixed point. Its equivalent contrapositive form has biological implications. Our goal is to provide an extension of Shih-Dong's theorem into all finite distributive lattices. Our method of proof is based on Shih-Dong's "collective effect method" as well as G. Birkhoff's representation theorem for finite distributive lattices.

Keywords. Discrete dynamical system, Finite distributive lattice, Fixed point, Generalized Boolean Jacobian matrix, Negative circuit, Positive circuit

Contents

1 Introduction

The Jacobian Conjecture is a long-standing problem in algebraic geometry. It was first stated in 1939 by O. H. Keller [5].

Conjecture 1.1 (The Jacobian Conjecture) *Suppose* $F : \mathbb{C}^n \to \mathbb{C}^n$ *is a polynomial map with the property that the derivative at each point is non-singular. Then must* F *be one-to-one.*

Here $F(z) = (f_1(z), \cdots, f_n(z))$, each f_i is a polynomial in n variables, and $z \in \mathbb{C}^n$. The Jacobian matrix of f at z is denoted by $F'(z)$. If F is indeed injective, then it is surjective and has an inverse which is a polynomial map. For an elementary proof of this see [11]. See the excellent survey [1] for the importance, background, and related results. S. Smale lists the Jacobian Conjecture as one of 18 great problems for 21 century [14].

In 1997, Cima, Gasull, and Mañosas [4] established a fixed point conjecture which is equivalent to the Jacobian Conjecture. It states that if $F : \mathbb{R}^n \to \mathbb{R}^n$ is a polynomial map with $\rho(F'(x)) < 1$ for each $x \in \mathbb{R}^n$, then F has a unique fixed point. Here $\rho(F'(x))$ denotes the spectral radius of $F'(x)$. In the study of automata networks, Shih and Ho [12] raised the Boolean counterpart conjecture of the fixed point conjecture. It states that if $F : \{0,1\}^n \to \{0,1\}^n$ is a Boolean map with $\rho(F'(x)) = 0$ for each $x \in \{0,1\}^n$, then F has a unique fixed point. Here $F'(x)$ is the Boolean Jacobian matrix of F at x and $\rho(F'(x))$ is the Boolean spectral radius of $F'(x)$, see [10] and [12]. In 2005, Shih and Dong used a collective effect method with an intricate argument to prove Shih-Ho's conjecture. Hence Shih-Dong's combinatorial theorem is stated as follows.

Theorem 1.2 (Shih-Dong) *Let* $F : \{0,1\}^n \to \{0,1\}^n$. *If* $\rho(F'(x)) = 0$ *for each* $x \in \{0,1\}^n$, *then* F *has a unique fixed point.*

Let us remark that every function $F : \{0,1\}^n \to \{0,1\}^n$, $F = (f_1, \cdots, f_n)$, is a Boolean polynomial, that is, each f_i is a Boolean polynomial. To see this, let $p_i : \{0,1\}^n \to \{0,1\}^n$ be defined by $p_1(t) = \bar{t}$, $p_2(t) = t$, and $\alpha_1 = 0$, $\alpha_2 = 1$. Then $p_j(\alpha_i) = 1$ if $i = j$, otherwise 0. Thus for $i = 1, \cdots, n$,

$$f_i(x_1, \cdots, x_n) = \sum_{j_1, \cdots, j_n \in \{1,2\}} p_{j_1}(x_1) \cdots p_{j_n}(x_n) f_i(\alpha_{j_1}, \cdots, \alpha_{j_n}).$$

The equivalent contrapositive form of Theorem 1.2 states that if $F : \{0,1\}^n \to \{0,1\}^n$ has multiple fixed points or has no fixed point, then there exists a point $x \in \{0,1\}^n$ such that the corresponding network $\Gamma(F'(x))$ has a circuit. This coincides with the Thomas conjecture which was stated in 1981 by the biologist René Thomas. When studying genetic regulatory networks, biologists often represent the results of their genetic and molecular investigations in terms of finite signed directed graphs. The vertices correspond to the members of the network (e.g., genes, RNA, proteins) and a positive (resp. negative) edge from i to j means that the member i activates (resp. represses) member j. These graphs are so called interaction graphs and biologists often use them as a basis to design dynamical models, using either a differential or a discrete framework [17].

In 1981, the biologist R. Thomas conjectured that: a necessary condition for multistationarity is that the interaction graph has a positive circuit and a necessary condition for sustained oscillations is that the interaction graph has a negative circuit, i.e. the sign of a circuit being defined as the product of the signs of its edges [16]. The multistationarity means that there exist several fixed points in the dynamics. Multistationarity is an important dynamical property since it is related to cell differentiation in biology [15, 16, 17, 18, 19].

Shih-Dong's theorem has been generalized and studied by Richard [7, 8, 9], Remy, Ruet, Thieffry [6], and C. Soulé [15]. The purpose of this paper is to provide an extension of Shih-Dong's theorem into all finite distributive lattices.

2 Definitions and Notations

This section is to present a conceptual framework for our generalized results. We state some notations and results concerning the spectra of Boolean matrices. The material can be found in the book by Robert [10].

Let $\{0,1\}$ be with three operations $+, \cdot, ^{-}$ defined as follows:

$$0+0=0\cdot 1=1\cdot 0=0\cdot 0=\bar{1}=0,\ 1+0=0+1=1+1=1\cdot 1=\bar{0}=1.$$

For $a,b\in\{0,1\}$, we usually suppress the dot "$\cdot$" of $a\cdot b$ and simply write ab. For $n\in\mathbb{N}$, let $\{0,1\}^n$ be the set of ordered n-tuples,

$$x=\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix},$$

with components $x_i\in\{0,1\}$ for each $i=1,\cdots,n$. We also write $x=(x_1,\cdots,x_n)$ interchangeably. The zero element of $\{0,1\}^n$ is the point $\mathbf{0}$, all of whose coordinates are 0. The order "$\leq$" in $\{0,1\}$ is given by $0\leq 0\leq 1\leq 1$. Thus for $a,b\in\{0,1\}$,

$$a+b=\max\{a,b\},\quad ab=\min\{a,b\}.$$

For $x,y\in\{0,1\}^n$, $x\leq y$ is meant that $x_i\leq y_i$ for each $i=1,\cdots,n$. For $x,y\in\{0,1\}^n$ and $\lambda\in\{0,1\}$, define

$$x+y=\begin{pmatrix} \max\{x_1,y_1\} \\ \vdots \\ \max\{x_n,y_n\} \end{pmatrix},\quad \lambda x=\begin{pmatrix} \min\{\lambda,x_1\} \\ \vdots \\ \min\{\lambda,x_n\} \end{pmatrix}.$$

A *Boolean matrix* is meant to be a matrix over $\{0,1\}$. Boolean matrix addition and Boolean matrix multiplication are the same as in the case of complex matrices but the concerned sums and products of entries are Boolean. Let $F:\{0,1\}^n\to\{0,1\}^n$ and let us write $F=(f_1,\cdots,f_n)$. According to Robert [10], the *incidence matrix* of F is the $n\times n$ Boolean matrix defined by $B(F)=(b_{ij})$, where $b_{ij}=0$ if f_i does not depend on x_j, $b_{ij}=1$ otherwise. Let A be an $n\times n$ Boolean matrix. A nonzero element

$u \in \{0,1\}^n$ is called a (Boolean) eigenvector of A if there exists $\lambda \in \{0,1\}$ such that $Au = \lambda u$; λ is called the (Boolean) eigenvalue associated with the eigenvector u. The symbol $\sigma(A)$ stands for the set of all (Boolean) eigenvalues of A, so that $\sigma(A) \subset \{0,1\}$. The *Boolean spectral radius* of A, which is denoted by $\rho(A)$, is defined to be the largest (Boolean) eigenvalue of A. Because $\sigma(A) \neq \emptyset$ (this fact is not a priori obvious, see[10]), $\rho(A) = 0$ or 1. Also $\rho(P^tAP) = \rho(A)$ for any permutation matrix P. For $x \in \{0,1\}^n$, let

$$\tilde{x}^j = \begin{pmatrix} x_1 \\ \vdots \\ \bar{x}_j \\ \vdots \\ x_n \end{pmatrix}.$$

The notation $\tilde{x}_i^j$ is meant that $\tilde{x}_i^j = \bar{x}^j$ if $i = j$, $\tilde{x}_i^j = x_i$ if $x \neq j$. For $x \in \{0,1\}^n$ and $\{j_1, \cdots, j_k\} \subset \{1, \cdots, n\}$, let us define $\tilde{x}^{j_1,\cdots,j_k} = y$ by

$$y_i = \begin{cases} x_i, & \text{if } i \neq j_1, \cdots, j_k, \\ \bar{x}_i, & \text{if } i = j_1, \cdots, j_k. \end{cases}$$

The *Boolean Jacobian matrix* of F at $x \in \{0,1\}^n$ is the (Boolean) $n \times n$ matrix defined by $F'(x) = (f_{ij}(x))$, where $f_{ij}(x) = 1$ if $f_i(x) \neq f_i(\tilde{x}^j)$, $f_{ij}(x) = 0$ otherwise. The discrete metric on $\{0,1\}$ is denoted by δ, that is, $\delta(x,y) = 1$ if $x \neq y$, $\delta(x,y) = 0$ if $x = y$. For $x, y \in \{0,1\}^n$, the *Boolean vector distance* $d(x,y)$ is defined by

$$d(x,y) = \begin{pmatrix} \delta(x_1, y_1) \\ \vdots \\ \delta(x_n, y_n) \end{pmatrix}.$$

Let us recall that the Boolean vector distance d satisfies:

$$\begin{aligned} &\text{(i)} \quad d(x,y) = d(y,x) \ (x, y \in \{0,1\}^n), \\ &\text{(ii)} \quad d(x,y) = \mathbf{0} \Leftrightarrow x = y \ (x, y \in \{0,1\}^n), \\ &\text{(iii)} \quad d(x,y) \leq d(x,z) + d(z,y) \ (x, y, z \in \{0,1\}^n), \end{aligned}$$

where $d(x,z) + d(z,y)$ is the Boolean sum in $\{0,1\}^n$.

The *digraph* (*directed graph*) of an $n \times n$ Boolean matrix $A = (a_{ij})$, denoted by $\Gamma(A)$, is the digraph having the vertex set $\{1, \cdots, n\}$ and a directed edge (j, i) from j to i if $a_{ij} = 1$. A *directed path* in $\Gamma(A)$ is a sequence of directed edges $i_1 i_2, i_2 i_3, \cdots$ in $\Gamma(A)$. A *circuit* $C = [c_1, \ldots, c_k]$ in a directed graph is a sequence of vertices which have edges from c_i to c_{i+1} for each $i = 1, \ldots, k$ and $c_1 = c_k$.

We now state some basic results concerning the spectral theory of Boolean matrices.

Theorem 2.1 *The following conditions are mutually equivalent:*

(i) $\rho(A) = 1$.
(ii) *A contains a principal submatrix which has no zero rows.*
(iii) *A contains a principal submatrix which has no zero columns.*
(iv) $\Gamma(A)$ *contains a circuit.*

Theorem 2.2 *The following conditions are mutually equivalent:*

(i) $\rho(A) = 0$.
(ii) *There exists a permutation matrix P such that* $P^t A P$ *is strictly upper triangular.*
(iii) *There exists a positive integer* $p \leq n$ *such that* $A^p = 0$.

For a finite lattice L, let us recall that the diagram of L is a graph whose vertices are elements of L and the edges correspond to the covering relation. If x covers y or y covers x then there is no $z \in L$ such that $y < z < x$ or $x < z < y$. We define the *metric* d_p on the diagram of L as follows. We say that $P = \{p^0, p^1, \cdots, p^k\}$ is a path of length k which connects p^0 and p^k if there has an edge between p^i and p^{i+1} in the diagram of L for each $i = 0, \cdots, k-1$. Denote by $l(P)$ the length of P. In particular, the singleton $\{x\}$ is a path from x to itself of length 0. For $x, y \in L$, define

$$d_p(x, y) = \min\{l(P) : P \text{ is a path connecting } x \text{ and } y\}.$$

Then d_p is a metric on L. Note that if $d_p(x, y) = 1$, then x covers y or y covers x.

Let us recall that an element a of a lattice is *join-irreducible* if $a = x \vee y$ implies that $a = x$ or $a = y$. We denote the set of join-irreducible elements of L as $\mathcal{J}(L) = \{a^1, \cdots, a^n\}$. For each $x \in L$, let $\eta(x) = \{a^i : a^i \leq x\}$ be the set of all join-irreducible elements of L which are less than or equal to x. Since L is finite, x is the join of $\eta(x)$; that is, every element of L can be obtained as a (possibly empty) join of a down-set of elements from $\{a^1, \cdots, a^n\}$.

For $x \in L$ and $i = 1, \cdots, n$, the ith switch $\tilde{x}^i$ of x is defined by

$$\tilde{x}^i = \begin{cases} \bigvee(\eta(x) \setminus \uparrow a^i) \text{ if } a^i \in \eta(x), \\ \bigvee(\eta(x) \cup \downarrow a^i) \text{ if } a^i \notin \eta(x), \end{cases}$$

where

$$\uparrow a^i = \{a \in L : \ a \text{ is a join-irreducible element and } a^i \leq a\},$$
$$\downarrow a^i = \{a \in L : \ a \text{ is a join-irreducible element and } a \leq a^i\}.$$

Given a mapping $F : L \to L$ and $x \in L$, let $F = (f_1, \cdots, f_n)$, where for each $i = 1, \cdots, n$, $f_i : L \to \{0_L, a^i\}$ is given by

$$f_i(x) = \begin{cases} a^i & \text{if } a^i \in \eta(F(x)), \\ 0_L & \text{if } a^i \notin \eta(F(x)). \end{cases}$$

Then $F(x) = \bigvee\{f_i(x) : i = 1, \cdots, n\}$ for each $x \in L$. Similarly, we denote $x = (x_1, \cdots, x_n)$ such that

$$x_i = \begin{cases} a^i & \text{if } a^i \in \eta(x), \\ 0_L & \text{if } a^i \notin \eta(x), \end{cases}$$

for each $i = 1, \cdots, n$, and $x = \bigvee\{x_i : i = 1, \cdots, n\}$.

We define the *generalized Boolean Jacobian matrix* of F at x as $F'(x) = (f_{ij}(x))$, where

$$f_{ij}(x) = \begin{cases} 1 & \text{if } f_i(x) \neq f_i(\tilde{x}^j) \text{ and } d_p(x, \tilde{x}^j) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma(F'(x))$ be the directed graph associated with $F'(x)$ having the vertex set $\{1, \cdots, n\}$ and a directed edge (j, i) from j to i if $f_{ij}(x) = 1$. The directed edge (j, i) is with *negative sign* if $x_j = 0_L$ and $f_i(x) = a^i$ or $x_j = a^j$ and $f_i(x) = 0_L$. The directed edge (j, i) is with *positive sign* if $x_j = 0_L$ and $f_i(x) = 0_L$ or $x_j = a^j$ and $f_i(x) = a^i$.

A *circuit* $C = [c_1, \cdots, c_k]$ in a directed graph is a sequence of vertices with $c_1 = c_k$ such that there exist an edge from c_i to c_{i+1} for each $i = 1, \cdots, k$. The *sign of a circuit* C is the product of the signs of its edges.

Let $I_F(x) = \{i : f_i(x) \neq x_i\}$. *The asynchronous dynamic graph of* F, denoted $G(F)$, is the directed graph whose set of vertices is L and whose set of edges is

$$\{(x, y) : x \in L, y = \tilde{x}^i \text{ for some } i \in I_F(x)\}.$$

Let us remark that $I_F(x) = \emptyset$ if and only if x is a fixed point of F.

A *trap domain of* $G(F)$ is a non-empty subset $D \subset L$ such that if any edge (x, y) of $G(F)$ satisfies $x \in D$, then $y \in D$. An *attractor of* $G(F)$ is the smallest trap domain. A *cyclic attractor* is an attractor of cardinality at least two. For a set U, denote by $|U|$ the cardinality of U.

According to the above definitions, Thomas' conjecture may be stated as follows.

Let L ba a finite distributive lattice and $F : L \to L$.

1. If F has multiple fixed points then $\bigcup_{x \in L} \Gamma(F'(x))$ has a positive circuit.

2. If $G(F)$ has a cyclic attractor then $\bigcup_{x \in L} \Gamma(F'(x))$ has a negative circuit.(In particular, if F has no fixed point then $\bigcup_{x \in L} \Gamma(F'(x))$ has a negative circuit.)

3 Fixed points and circuit-free

In this section, we extend Theorem 1.2 from the Boolean case to the finite distributive lattice. We shall establish the following:

Theorem 3.1 *Let L be a finite distributive lattice with $\mathfrak{J}(L) = \{a^1, \ldots, a^k\}$. If $F : L \to L$ is such that $\Gamma(F'(x))$ has no circuit for all $x \in L$, then there exists a unique fixed point.*

To prove the full generality we adopt the thinking of the Chinese proverb, "Many a little makes a mickle." Thus we need the notion of sublattice $[a, b]$ generated by $a, b \in L$, $a < b$. Let $a, b \in L$, $a < b$. We define

$$[a, b] = \{x \in L : a \leq x \leq b\}.$$

We call $[a, b]$ a sublattice generated by a, b. Let $I = \{i : a_i < b_i\}$. Then

$$[a, b] = \{x \in L : x_i = a_i = b_i \text{ for all } i \neq I\}.$$

Proposition 3.2 *Let L be a finite distributive lattice. For $a, b \in L$, $a < b$, the sublattice $[a, b]$ is a distributive lattice.*

Proof. For $x, y, z \in [a, b] \subset L$, L is distributive, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Since $[a, b] = \{x \in L : a \leq x \leq b\}$, we have $c \vee d, c \wedge d \in [a, b]$ for all $c, d \in [a, b]$. This shows $[a, b]$ is a distributive lattice. □

Let $a, b \in L$, $a < b$. We say that $\alpha \in [a, b]$ is a *local fixed point* if

$$f_i(\alpha) = \alpha_i, \ \forall i \in \{i : a_i < b_i\}.$$

Theorem 3.3 *Let L be a finite distributive lattice with $\mathfrak{J}(L) = \{a^1, \ldots, a^k\}$ and let $F : L \to L$ be such that $\Gamma(F'(x))$ has no circuit for all $x \in L$. Then for every $a, b \in L$ with $a < b$, there exists a unique local fixed point $\alpha \in [a, b]$.*

To prove Theorem 3.1, we apply Theorem 3.3 by taking $a = 0_L$, $b = 1_L$. Thus we have a generalization of Shih-Dong's theorem to all finite distributive lattices.

In the case $L = \{0,1\}^n$, Theorem 3.3 reduces to a lemma proved by Shih and Dong [13]. The proof of Theorem 3.3 reveals a phenomenon that a global feature of a system emerges from collective behavior of its many components. In terms of the Chinese proverb quoted earlier, Theorem 3.3 is many a little and Theorem 3.1 is the mickle.

To establish Theorem 3.3, we need the following lemmas. The proof of Theorem 3.3 is modelled after the proof of Shih-Dong's theorem.

Lemma 3.4 *Let L be a finite distributive lattice with $\mathfrak{J}(L) = \{a^1, \ldots, a^k\}$. Then for each $x, y \in L$ with $x < y$, the set $I = \{i : x_i < y_i\}$ is non-empty and $y = \tilde{x}^i$ for all $i \in I$ when $d_p(x, y) = 1$.*

Proof. For $x < y$, if $x_i = a^i$, then $a^i \leq x < y$ and hence $y_i = 1$. This shows $x_i \leq y_i$ for each i. If $I = \emptyset$, then $x_i = y_i$ for each i. We have $x = y$, a contradiction.

If $d_p(x, y) = 1$, then y covers x. For $i \in I$, since $x_i = 0_L$, $y_i = a^i$ and $\tilde{x}^i_i = a^i$, we have $x < \tilde{x}^i \leq y$ by the definition of $\tilde{x}^i$. Thus $\tilde{x}^i = y$ for each $i \in I$. □

The collection of down-sets of any partially ordered set forms a lattice in which the lattice's partial ordering is given by the set inclusion. The join operation corresponds to the set union, and the meet operation corresponds to the set intersection. Since the set union and the set intersection obey the distributive law, the collection of down-sets is a distributive lattice. Birkhoff's representation theorem asserts that any finite distributive lattice L is isomorphic to the lattice of down-sets of the join-irreducible elements of L [2, 3].

According to Birkhoff's representation theorem, we can prove the following lemma.

Lemma 3.5 *Let L be a finite lattice. Then L is distributive if and only if $d_p(x, y) = |\eta(x \vee y)| - |\eta(x \wedge y)|$ for all $x, y \in L$.*

Proof. We first show that the inequality $d_p(x, y) \leq |\eta(x \vee y)| - |\eta(x \wedge y)|$ holds for all $x, y \in L$ where L is any finite lattice (not necessarily distributive). Let $x, y \in L$. If $x \leq y$, then it is readily seen that $d_p(x, y) \leq |\eta(x \vee y)| - |\eta(x \wedge y)|$ by the definition of $d_p(x, y)$ and Lemma 3.4.

Let $J = \{i : x_i \neq y_i\}$, $A = \{i : (x \wedge y)_i < x_i\}$, and $B = \{i : (x \wedge y)_i < y_i\}$. It is obvious that $A \subset J$, $B \subset J$, and $J \subset A \cup B$. If $i \in A$ and $i \in B$, then $x_i = y_i = a^i$. It implies that $(x \wedge y)_i = a^i$ and $i \notin A$, which is a contradiction. It follows that $A \cap B = \emptyset$ and $A \cup B = J$. Thus

$$d_p(x, y) \leq d_p(x, x \wedge y) + d_p(y, x \wedge y) \leq |A| + |B| = |J|.$$

Let $i \in J$. Then $a^i \in \eta(x \vee y)$ and $a^i \notin \eta(x \wedge y)$. Thus

$$|J| = |\eta(x \vee y)| - |\eta(x \wedge y)|.$$

We have shown that $d_p(x, y) \leq |\eta(x \vee y)| - |\eta(x \wedge y)|$ holds for any x, y in arbitrary finite lattice L.

Now we assume that L is distributive. According to Birkhoffs representation theorem, we have the following triangle inequality:

$$|\eta(x \vee y)| - |\eta(x \wedge y)| \leq |\eta(x \vee z)| - |\eta(x \wedge z)| + |\eta(y \vee z)| - |\eta(y \wedge z)|$$

for all $x, y, z \in L$.

Let $x, y \in L$. In case $d_p(x, y) = 1$, we may assume that $x < y$. Suppose that $|\eta(x \vee y)| - |\eta(x \wedge y)| \geq 2$, then there exist $a^{i_1}, a^{i_2} \in \mathcal{J}(L)$ such that $x_i = 0_L$ and $y_i = a^i$ for $i = i_1, i_2$. If a^{i_1} and a^{i_2} are incomparable, that is $a^{i_1} \nleq a^{i_2}$ and $a^{i_2} \nleq a^{i_1}$, then $x < \tilde{x}^{i_1} < y$ by Birkhoff's representation theorem. It contradicts $d_p(x, y) = 1$. If $a^{i_1} < a^{i_2}$ or $a^{i_2} < a^{i_1}$, then $x < \tilde{x}^{i_1} < y$ or $x < \tilde{x}^{i_2} < y$. It also contradicts $d_p(x, y) = 1$ and hence $d_p(x, y) = 1 = |\eta(x \vee y)| - |\eta(x \wedge y)|$.

In case $d_p(x, y) = k$, there exists a path $\{x = p^0, p^1, \ldots, p^k = y\}$ such that $d_p(p^i, p^{i+1}) = 1 = |\eta(p^i \vee p^{i+1})| - |\eta(p^i \wedge p^{i+1})|$ for each $i = 0, \ldots, k-1$. Thus

$$\begin{aligned}
|\eta(x \vee y)| - |\eta(x \wedge y)| &\leq |\eta(x \vee p^1)| - |\eta(x \wedge p^1)| + |\eta(p^1 \vee p^2)| - |\eta(p^1 \wedge p^2)| \\
&\quad + \cdots + |\eta(p^{k-1} \vee y)| - |\eta(p^{k-1} \wedge y)| \\
&= d_p(x, p^1) + d_p(p^1, p^2) + \cdots + d_p(p^{k-1}, y) \\
&= k = d_p(x, y).
\end{aligned}$$

It follows that $d_p(x, y) = |\eta(x \vee y)| - |\eta(x \wedge y)|$.

Conversely, let $d_p(x, y) = |\eta(x \vee y)| - |\eta(x \wedge y)|$ for all $x, y \in L$. If L is not distributive, then L has a sublattice isomorphic to M_3 (the *diamond*) or

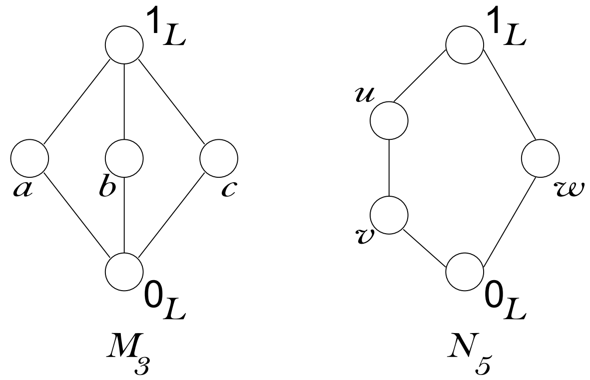

Figure 1: The diamond lattice M_3 (left) and the pentagon lattice N_5 (right).

N_5 (the *pentagon*), as shown in Figure 1. It is obvious that $2 = d_p(0_L, 1_L) < |\eta(1_L)| - |\eta(0_L)| = 3$ when L is M_3 or N_5.

If M is a sublattice of L and M is isomorphic to M_3, then there exist $a, b, c \in M$ such that $0_M < a, b, c < 1_M$ and $a \vee b = b \vee c = a \vee c = 1_M$. Let $L' = [\bigwedge M, \bigvee M] = \{x \in L : \bigwedge M \leq x \leq \bigvee M\}$ be a sublattice of L and $I = \{i : (\bigwedge M)_i < (\bigvee M)_i\} = \{i_1, i_2, \ldots, i_k\}$. Since $a \vee b = b \vee c = a \vee c = 1_M$, there exist $i_a, i_b, i_c \in I$ such that if $x \in M$ with $x_i, x_j \neq 0_L$ for $i, j \in \{i_a, i_b, i_c\}$, then $x = \bigvee M$.

Since $d_p(\bigwedge M, \bigwedge M) = |\eta(\bigvee M)| - |\eta(\bigwedge M)| = |I|$, there exists a path $P = \{\bigwedge M = z^0, z^1, \ldots, z^k = \bigvee M\}$ where $d_p(z^j, z^{j-1}) = 1$ for each $j = 1, \ldots, k$. By Lemma 3.4, we have $z^j = z^{j-1} \vee a^{i_j}$ for some $i_j \in I$ for each $j = 1, \ldots, k$. However the condition "if $x \in M$ with $x_i, x_j \neq 0_L$ for $i, j \in \{i_a, i_b, i_c\}$, then $x = \bigvee M$" implies that $z^j = z^{j+1}$ for some $j < k$ by pigeonhole principle, which is a contradiction.

On the other hand, if N is a sublattice of L and N is isomorphic to N_5, then there exist $u, v, w \in M$ such that $0_N < u, v, w < 1_N$ and $u \vee w = v \vee w = 1_N$. Let $L'' = [\bigwedge N, \bigvee N] = \{x \in L : \bigwedge N \leq x \leq \bigvee N\}$ be a sublattice of L and $I = \{i : (\bigwedge N)_i < (\bigvee N)_i\} = \{i_1, i_2, \ldots, i_k\}$. Since $u \vee w = v \vee w = 1_N$, there exist $i_u, i_v, i_w \in I$ such that if $x \in M$ with $x_w, x_i \neq 0_L$ for $i \in \{i_u, i_v\}$, then $x = \bigvee N$. Let $I_1 = \{i : (\bigwedge N)_i < w_i\}$ and $I_2 = \{i : w_i < (\bigvee N)_i\}$. Then we have $I_1 \cap I_2 = \emptyset$ and $I_1 \cup I_2 = I$. In particular, we have $i_u, i_v \in I_2$. Let $l = |I_1|$, without loss of generality, we may assume that $i_j \in I_1$ for $j = 1 \cdots, l$ and $i_j \in I_2$ for $j = l+1, \ldots, k$; this is possible by renumbering the join-irreducible elements a^i, $i \in I$. Since $d_p(\bigwedge N, \bigwedge N) = |\eta(\bigvee N)| - |\eta(\bigwedge N)| = |I|$, there exists a path $P = \{\bigwedge N = z^0, \ldots, z^{l-1}, z^l = w, z^{l+1}, \ldots, z^k = \bigvee N\}$ such that $d_p(z^j, z^{j-1}) = 1$ for each $j = 1, \ldots, k$. By Lemma 3.4, we have $z^j = z^{j-1} \vee a^{i_j}$ for some $i_j \in I$ for each $j = 1, \ldots, k$. However the condition "if $x_w, x_i \neq 0_L$ for $i \in \{i_u, i_v\}$, then $x = \bigvee N$" implies that $z^j = \bigvee N$ for some $j \in I_2$ and $j < k$ by pigeonhole principle, which is a contradiction. This completes the proof. □

Lemma 3.6 *Let L be a finite distributive lattice and $x, y \in L$ with $x \neq y$. Then there exists a join-irreducible element a^i such that $x_i \neq y_i$ and $d_p(x, \tilde{x}^i) = 1$.*

Proof. Let $x, y \in L$ and $x \neq y$. Since $x \neq y$, there exists j such that $x_j \neq y_j$. We may assume that $x_j = 0_L$ and $y_j = a^j$. If $d_p(x, \tilde{x}^j) = 1$, then we set $i = j$. If $d_p(x, \tilde{x}^j) \geq 2$, by the definition of $\tilde{x}^j$ and Birkhoff's representation theorem, then there exists a join-irreducible element a^k such that $x < \tilde{x}^k < \tilde{x}^j$. We have $x \wedge \tilde{x}^k = x$, $x \wedge \tilde{x}^j = x$, $x \vee \tilde{x}^k = \tilde{x}^k$, and $x \vee \tilde{x}^j = \tilde{x}^j$. It implies that $d_p(x, \tilde{x}^k) < d_p(x, \tilde{x}^j)$ by Lemma 3.5 and $x_k = 0_L$, $y_k = a^k$ (because $y_j = a^j$). Repeating the above process, we get i such that $x_i \neq y_i$ and $d_p(x, \tilde{x}^i) = 1$. □

We proceed now to prove Theorem 3.3.

We prove the assertion by induction on $d_p(a, b)$. If $d_p(a, b) = 1$, set $b = \tilde{a}^i$, then the hypothesis $\Gamma(F'(a)) = 0$ implies that $f_{ii}(a) = 0$ and hence $(F(a))_i = (F(\tilde{a}^i))_i = (F(b))_i$. So exactly one of the statements $(F(a))_i = a_i$, $(F(b))_i = b_i$ is true. Thus the assertion is valid.

We now assume that the theorem holds for all $a', b' \in L$ such that $1 \leq d_p(a', b') < d_p(a, b)$. We first settle the uniqueness question. Suppose, on the contrary, that α and β are two distinct points in the sublattice $[a, b]$ such that

$$(F(\alpha))_i = \alpha_i \text{ and } (F(\beta))_i = \beta_i \text{ for all } i \in I = \{i : a_i < b_i\}.$$

Case 1. $d_p(\alpha, \beta) < d_p(a, b)$. Then $\alpha, \beta \in [\alpha \wedge \beta, \alpha \vee \beta] \subset [a, b]$ and

$$(F(\alpha))_i = \alpha_i \text{ and } (F(\beta))_i = \beta_i \text{ for all } i \in \{i : (\alpha \wedge \beta)_i < (\alpha \vee \beta)_i\},$$

in contradiction to the uniqueness assertion of the induction hypothesis.

Case 2. $d_p(\alpha,\beta) = d_p(a,b)$. It implies that $\alpha_i \neq \beta_i$ for all $i \in I$. Let $I_\beta = \{i \in I : d_p(\beta,\tilde{\beta}^i) = 1\}$. For $i \in I_\beta$, $d_p(\alpha,\tilde{\beta}^i) < d_p(\alpha,\beta)$ and $\alpha,\tilde{\beta}^i \in [\alpha \wedge \tilde{\beta}^i, \alpha \vee \tilde{\beta}^i]$. By the induction hypothesis, there exists $j \in I \setminus \{i\}$ such that $(F(\tilde{\beta}^i))_j \neq \tilde{\beta}^i_j$. Because $d_p(\beta,\tilde{\beta}^i) = 1$, we have $\beta_j = \tilde{\beta}^i_j$ and hence $\tilde{\beta}^i_j = (F(\beta))_j$. By Lemma 3.6, there exists $l \in I_\beta$ such that $(F(\beta))_l \neq (F(\tilde{\beta}^i))_l$. Thus $F'(\beta)$ contains a principal submatrix

$$\begin{array}{c} \\ i_1 \\ \vdots \\ i_m \end{array} \begin{array}{c} \begin{array}{ccc} i_1 & \cdots & i_m \end{array} \\ \begin{pmatrix} * & \cdots & * \\ \vdots & \ddots & \vdots \\ ** & \cdots & * \end{pmatrix} \end{array}$$

of order m which has no zero columns where $I_\beta = \{i_1,\ldots,i_m\}$ and the symbol $*$ denotes the entry possibly 1. We conclude that $\rho(F'(\beta)) = 1$, contrary to the spectral condition. We now arrive at a contradiction for Cases 1 and 2. This contradiction completes the proof of the uniqueness part.

Now we prove the existence part. By Lemma 3.6, there exists $l \in I$ such that $a_l = 0$, $b_l = 1$ and $d_p(a,\tilde{a}^l) = 1$.

Case 1. $d_p(b,\tilde{b}^l) = 1$. According to the induction hypothesis, there exist a unique fixed point $x \in [a,\tilde{b}^l]$ and a unique fixed point $y \in [\tilde{a}^l, b]$ such that

$$(F(x))_i = x_i \text{ and } (F(y))_i = y_i \text{ for all } i \in I \setminus l. \tag{3.1}$$

We want to use two distinct points x and y to synthesis a point α that satisfies the theorem; it has to split the arguments into two cases.

Case 1.1. $d_p(x,y) < d_p(a,b)$. Then $[x\wedge y, x\vee y] \subset [a,b]$ and $d(x\wedge y, x\vee y) = d_p(x,y) < d_p(a,b)$. By the induction hypothesis, there exists a unique $z \in [x\wedge y, x\vee y]$ such that

$$(F(z))_i = z_i \text{ for all } i \in \{i : (x\wedge y)_i < (x\vee y)_i\}. \tag{3.2}$$

Since $x_l = 0$ and $y_l = 1$, $l \in \{i : (x\wedge y)_i < (x\vee y)_i\}$. If $z_l = 0$, then

$$x, z \in [x\wedge z, x\vee z] \subset [x\wedge y, x\vee y].$$

By (3.1), (3.2), and the uniqueness assertion of the induction hypothesis, we have $x = z$. Thus x satisfies the equations

$$(F(x))_i = x_i \text{ for all } i \in I.$$

On the other hand, if $z_l = 1$ then $y = z$. Hence the existence assertion of the theorem follows if we take $\alpha = x$ or $\alpha = y$.

Case 1.2. $d_p(x, y) = d_p(a, b)$. Then $x_i \neq y_i$ for all $i \in I$. By the condition $\rho(F'(y)) = 0$, it follows that there exists an $j \neq l$ with $j \in I_y$ where $I_y = \{i \in I : d_p(y, \tilde{y}^i) = 1\}$ such that

$$(F(\tilde{y}^j))_i = (F(y))_i \text{ for all } i \in I_y. \tag{3.3}$$

If $j = l$, then $\tilde{y}^l_l = 0$ and

$$(F(\tilde{y}^l))_i = (F(y))_i \text{ for all } i \in I_y. \tag{3.4}$$

By (3.1), (3.4), and Lemma 3.6, we have

$$(F(\tilde{y}^l))_i = (F(y))_i = y_i = \tilde{y}^l_i \text{ for all } i \in I \setminus \{l\}.$$

Thus x and $\tilde{y}^l$ would be two distinct points in $[a, \tilde{b}^l]$ such that $(F(x))_i = x_i$ and $(F(\tilde{y}^l))_i = \tilde{y}^l_i$ for all $i \in I \setminus \{l\}$, contrary to the uniqueness assertion of the induction hypothesis. By (3.3) and Lemma 3.6, we have

$$(F(\tilde{y}^j))_i = (F(y))_i \text{ for all } i \in I. \tag{3.5}$$

Since $x_j = \tilde{y}^j_j$, we have $d_p(x, \tilde{y}^j) = d_p(x, y) - 1 < d_p(x, y)$. By the induction hypothesis there exists a unique point $w \in [x \wedge \tilde{y}^j, x \vee \tilde{y}^j]$ with

$$(F(w))_i = w_i \text{ for all } i \in \{i : (x \wedge \tilde{y}^j)_i < (x \vee \tilde{y}^j)_i\}. \tag{3.6}$$

If $w_l = \tilde{y}^j_l$, then

$$w, \tilde{y}^j \in [w \wedge \tilde{y}^j, w \vee \tilde{y}^j] \subset [x \wedge \tilde{y}^j, x \vee \tilde{y}^j].$$

By (3.1), (3.5), (3.6), and the uniqueness assertion of the induction hypothesis we have $w = \tilde{y}^j$. By (3.3),

$$(F(y))_l = (F(\tilde{y}^j))_l = \tilde{y}^j_l = y_l.$$

Thus

$$(F(y))_i = y_i \text{ for all } i \in I.$$

If $w_l = x_l$, then

$$w, x \in [w \wedge x, w \vee x] \subset [x \wedge \tilde{y}^j, x \vee \tilde{y}^j].$$

By (3.1), (3.6), and the uniqueness assertion of the induction hypothesis we have $w = x$ and hence

$$(F(x))_l = x_l.$$

We have

$$(F(x))_i = x_i \text{ for all } i \in I.$$

The existence assertion of the theorem follows if we take $\alpha = x$ or $\alpha = y$.
Case 2. $d_p(b, \tilde{b}^l) \geq 2$ for all $l \in \{i \in I : d_p(a, \tilde{a}^i) = 1\}$. Let $i_1 \in I$ such that $d_p(a, \tilde{a}^{i_1}) = 1$. By the definition of $d_p(b, \tilde{b}^{i_1})$ and the definition of $\tilde{b}^{i_1}$, there exists $a^{i_2} \in \mathfrak{J}(L)$ such that $a^{i_1} < a^{i_2}$. According to Lemma 3.6, we can assume that $d_p(b, \tilde{b}^{i_2}) = 1$.

According to the induction hypothesis, there exist a unique fixed point $x \in [a, \tilde{b}^{i_2}]$ and a unique fixed point $y \in [\tilde{a}^{i_1}, b]$ such that

$$(F(x))_i = x_i \text{ for all } i \in I \setminus i_2 \text{ and } (F(y))_i = y_i \text{ for all } i \in I \setminus i_1. \tag{3.7}$$

Notice that $x_{i_2} = 0$ and $y_{i_1} = 1$.

Similarly, we want to use two points x and y to synthesis a point α that satisfies the theorem; it has to split the arguments into two cases.

Case 2.1. $x_{i_1} = 0$ or $y_{i_2} = 1$. If $(F(x))_{i_2} \neq x_{i_2}$ or $(F(y))_{i_1} \neq y_{i_1}$, then

$$(F(x))_{i_2} = 1 \text{ or } (F(y))_{i_1} = 0.$$

Since $a^{i_1} < a^{i_2}$, we have $(F(x))_{i_1} = 1$ or $(F(y))_{i_2} = 0$. It contradicts (3.7). Thus $x_{i_1} = 0$ or $y_{i_2} = 1$ implies that $(F(x))_{i_2} = x_{i_2}$ or $(F(y))_{i_1} = y_{i_1}$ and

$$(F(x))_i = x_i \text{ or } (F(y))_i = y_i \text{ for all } i \in I.$$

The existence assertion of the theorem follows if we take $\alpha = x$ or $\alpha = y$.

Case 2.2. $x_{i_1} = 1$ and $y_{i_2} = 0$. Thus $x_i = y_i$ for $i = i_1, i_2$, we have $d_p(x, y) < d_p(a, b)$ and $i_1, i_2 \notin \{i : (x \wedge y)_i < (x \vee y_i\}$. By the induction hypothesis there exists a unique point $z \in [x \wedge y, x \vee y]$ such that

$$(F(z))_i = z_i \text{ for all } i \in \{i : (x \wedge y)_i < (x \vee y_i)\}.$$

According to (3.7) and the induction hypothesis, it implies that $x = y$ and hence

$$(F(x))_i = x_i \text{ for all } i \in I.$$

Hence the existence assertion of the theorem follows if we take $\alpha = x = y$. This completes the inductive proof of Theorem 3.3.

4 Fixed points and positive circuits

In this section, we prove the first Thomas conjecture. It is stated that if $F : L \to L$ has more than one fixed points where L is a finite distributive lattice, then $\bigcup_{x \in L} \Gamma(F'(x))$ has a positive circuit. In fact, we prove the following statement. If $F : L \to L$ has more than one fixed points where L is a finite distributive lattice, then there exists a point $x \in L$ such that $\Gamma(F'(x))$ has a positive circuit. We shall establish the following:

Theorem 4.1 *Let L be a finite distributive lattice and $F : L \to L$. If $\Gamma(F'(x))$ has no positive circuit for each $x \in L$, then for each $a, b \in L$ with $a < b$, F has at most 1 local fixed point in $[a, b]$.*

Proof. We prove the assertion by induction on the distance $d_p(a, b)$. The case of $d_p(a, b) = 1$, let $b = \tilde{a}^i$ by Lemma 3.6. By hypothesis $\rho(F'(a)) = 0$, it implies that $f_{ii}(a) = 0$ and hence $f_i(a) = f_i(\tilde{a}^i) = f_i(b)$. So exactly one of the statements $f_i(a) = a_i$, $f_i(b) = b_i$ is true. Thus the theorem holds for the case of $d_p(a, b) = 1$.

We may assume that the theorem holds for all $a', b' \in L$ such that $1 \leq d_p(a', b') < d_p(a, b)$. Suppose, on the contrary, that α and β are two distinct points in the sublattice $[a, b]$ such that

$$f_i(\alpha) = \alpha_i \text{ and } f_i(\beta) = \beta_i \text{ for all } i \in I = \{i : a_i < b_i\}.$$

Notice that $[a, b]$ is a distributive lattice. For $x, y, z \in [a, b] \subset L$, L is distributive, we have

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Since $[a, b] = \{x \in L : a \leq x \leq b\}$, we have $c \vee d, c \wedge d \in [a, b]$ for all $c, d \in [a, b]$. This shows $[a, b]$ is a distributive lattice.

Case 1. $d_p(\alpha, \beta) < d_p(a, b)$. We have $\alpha, \beta \in [\alpha \wedge \beta, \alpha \vee \beta] \subset [a, b]$. Since $[a, b]$ is distributive, by Lemma 3.5,

$$d_p(\alpha \vee \beta, \alpha \wedge \beta) = |\eta(\alpha \vee \beta)| - |\eta(\alpha \wedge \beta)| = d_p(\alpha, \beta) < d_p(a, b).$$

However, $f_i(\alpha) = \alpha_i$ and $f_i(\beta) = \beta_i$ for all $i \in \{i : (\alpha \wedge \beta)_i < (\alpha \vee \beta)_i\}$, in contradiction to the induction hypothesis.

Case 2. $d_p(\alpha, \beta) = d_p(a, b)$. Let $I_\alpha = \{i \in I : d_p(\alpha, \tilde{\alpha}^i) = 1\}$. Then $I_\alpha \neq \emptyset$ by Lemma 3.6. For each $i \in I_\alpha$, since $d_p(a, b) > 1$, we have $\tilde{\alpha}^i \in [\tilde{\alpha}^i \wedge \beta, \tilde{\alpha}^i \vee \beta]$ and $\tilde{\alpha}^i \neq \beta$. Since $[\tilde{\alpha}^i \wedge \beta, \tilde{\alpha}^i \vee \beta]$ is distributive, by Lemma 3.5,

$$\begin{aligned} d_p(\tilde{\alpha}^i \vee \beta, \tilde{\alpha}^i \wedge \beta) &= |\eta(\tilde{\alpha}^i \vee \beta)| - |\eta(\tilde{\alpha}^i \wedge \beta)| = d_p(\tilde{\alpha}^i, \beta) \\ &= d_p(\alpha, \beta) - 1 < d_p(a, b). \end{aligned}$$

The induction hypothesis implies that $\tilde{\alpha}^i$ is not a local fixed point in $[\tilde{\alpha}^i \wedge \beta, \tilde{\alpha}^i \vee \beta]$. There exists $j' \in I \setminus \{i\}$ such that $f_{j'}(\tilde{\alpha}^i) \neq \tilde{\alpha}^i_{j'} = \alpha_{j'}$. Let $z = (a \vee F(\tilde{\alpha}^i)) \wedge b$. Then $z \in [a, b]$ and $z_{i'} = f_{i'}(\tilde{\alpha}^i)$ for each $i' \in I$ and $\alpha \neq z$. By Lemma 3.6, there exists $j \in I_\alpha$ such that $\alpha_j \neq z_j = f_j(\tilde{\alpha}^i)$. Thus $f_{ij}(\alpha) = 1$. Let $F'(\alpha)|_{I_\alpha} = (f_{ij}(\alpha)|_{I_\alpha})$ be the principal submatrix of $F'(\alpha)$ where $f_{ij}(\alpha)|_{I_\alpha} = f_{ij}(\alpha)$ if $i, j \in I_\alpha$; otherwise $f_{ij}(x)|_{I_\alpha} = 0$. Then $F'(\alpha)$ contains a principal submatrix which has no zero columns. It follows that $F'(\alpha)|_{I_\alpha}$ has a circuit $C = [c_1, \ldots, c_k, c_{k+1}]$ [10, 13]. We have

$$f_{c_{i+1}}(\tilde{\alpha}^{c_i}) \neq f_{c_{i+1}}(\alpha) = \alpha_{c_{i+1}} = \sigma_i(\alpha_{c_i}) \text{ for each } i = 1, \ldots k,$$

where $\sigma_i : \{0_L, a^{c_i}\} \to \{0_L, a^{c_{i+1}}\}$ satisfies $\sigma_i(0_L) = a^{c_{i+1}}$ and $\sigma_i(a^{c_i}) = 0_L$ when the edge from c_i to c_{i+1} is negative, or satisfies $\sigma_i(0_L) = 0_L$ and $\sigma_i(a^{c_i}) = a^{c_{i+1}}$ when the edge from c_i to c_{i+1} is positive. Since

$$\alpha_{c_1} = \alpha_{c_{k+1}} = (\sigma_k \circ \cdots \circ \sigma_1)(\alpha_{c_1}),$$

the sign of the circuit C is positive, which is a contradiction. This completes the proof. □

By taking $a = 0_L$ and $b = 1_L$ in Theorem 4.1, we have

Theorem 4.2 *Let L be a finite distributive lattice and $F : L \to L$. If $\Gamma(F'(x))$ has no positive circuit for each $x \in L$, then F has at most one fixed point.*

In the following, we state explicitly the contrapositive form of Theorem 4.2.

Theorem 4.3 *Let L be a finite distributive lattice. If $F : L \to L$ has more than one fixed points, then there exists a point $x \in L$ such that $\Gamma(F'(x))$ has a positive circuit.*

5 Fixed points and negative circuits

In this section, we prove the second Thomas conjecture. It is stated as follows.

Theorem 5.1 *Let L be a finite distributive lattice and $F : L \to L$. Let D be a cyclic attractor of $G(F)$. Then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit.*

As an immediate consequence of Theorem 5.1, we have

Theorem 5.2 *Let L be a finite distributive lattice. If $F : L \to L$ has no fixed point, then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit.*

Proof. If F has no fixed point, then $G(F)$ contains a cyclic attractor. By Theorem 5.1, $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit. □

To prove Theorem 5.1, we need the following lemmas.

Lemma 5.3 *Let L be a finite lattice and D be an attractor. For $x, y \in D$, there exists a path from x to y in $G(F)$.*

Proof. Suppose that there exist $x, y \in D$ such that there has no path from x to y in $G(F)$. Let $D' = \{z :$ there exists a path from x to z in $G(F)\}$. Then D' is a trap domain and $y \notin D'$. So $D' \subsetneq D$, which contradicts the smallest trap domain D. □

Lemma 5.4 *Let L be a finite distributive lattice. If $(x, y) \in G(F)$, then $\{i : x_i \neq y_i\} \subset I_F(x)$.*

Proof. If $(x, y) \in G(F)$, then there exists $i \in I_F(x)$ such that $y = \tilde{x}^i$. We may assume that $x < y$. Let $j \in \{i : x_i \neq y_i\}$, we have $x_j < y_j$.
Case 1. $a^i < a^j$. According to Birkhoff's representation theorem, $y_j = (\tilde{x}^i)_j = 0_L = x_j$, which is a contradiction.
Case 2. a^i and a^j are incomparable. According to Birkhoff's representation theorem, $y_j = (\tilde{x}^i)_j = 0_L = x_j$, which is a contradiction.
Case 3. $a^j < a^i$. Since $f_i(x) = a^i$, it implies that

$$a^j < a^i \leq F(x) = \bigvee \{f_i(x) : i = 1, \ldots, n\}.$$

Thus $f_j(x) = a^j \neq x_j$, $j \in I_F(x)$. □

Let G_1 and G_2 be two directed graphs with the same set of vertices V and with set of edges A_1 and A_2 respectively. We say that G_1 is a *subgraph* of G_2 if $A_1 \subset A_2$. For $x \in L$, we set

$$s_i(x) = \begin{cases} 1 & \text{if } f_i(x) > x_i, \\ 0 & \text{if } f_i(x) = x_i, \\ -1 & \text{if } f_i(x) < x_i. \end{cases}$$

Lemma 5.5 *Let L be a finite distributive lattice and $F : L \to L$. Let $\{x^0, x^1, \ldots, x^k\}$ be a path of $G(F)$ of length $k \geq 1$, and let $i \in I_F(x^k)$. If $s_i(x^p) \neq s_i(x^k)$ for all $0 \leq p < k$, then there exists $j \in I_F(x^0)$ such that $\bigcup_{x \in L} \Gamma(F'(x))$ has a path from j to i with sign $s_j(x^0)s_i(x^k)$.*

Proof. We prove the assertion by the induction on the length of the path k. In case $k = 1$, by hypothesis, $s_i(x^0) \neq s_i(x^1)$ and $x^1 = \widetilde{x^0}^l$ for some $l \in I_F(x^0)$.
Case 1. $d_p(x^0, x^1) = 1$. We set $l = j$.

Case 1.1. $i \neq j$. Then $x_i^0 = x_i^1$. We may assume that $s_i(x^1) = 1$. Thus $f_i(\widetilde{x^0}^j) = f_i(x^1) > x_i^1 = x_i^0 \geq f_i(x^0)$, it implies that $f_{ij}(x^0) = 1$. If $x_j^0 = 0_L$, then $s_j(x^0) = 1$ and $x_j^0 = f_i(x^0) = 0_L$. It follows that $\Gamma(F'(x^0))$ has an edge from j to i with positive sign. If $x_j^0 = a^j$, then $s_j(x^0) = -1$. Since $f_i(x^0) = 0_L$, $\Gamma(F'(x^0))$ has an edge from j to i with negative sign. Thus $\Gamma(F'(x^0))$ has an edge from j to i with sign $s_j(x^0)s_i(x^k)$ when $i \neq j$.

Case 1.2. $i = j$. Then $s_i(x^0)s_i(x^1) = -1$. We may assume $s_i(x^1) = 1$. Thus $f_i(\widetilde{x^0}^i) = f_i(x^1) > x_i^1 = f_i(x^0)$ and $x_i^0 \neq f_i(x^0)$, it implies that $f_{ii}(x^0) = 1$ and either $x_i^0 = 0_L$ or $f_i(x^0) = 0_L$. Thus $G(F'(x^0))$ has an edge from j to i with negative sign. This completes the proof of Case 1.
Case 2. $d_p(x^0, x^1) > 1$. Let $\{x^0 = p^0, p^1, \cdots, p^m = x^1\}$ be a path which connects x^0 and x^1 with $p^r < p^{r+1}$ or $p^r > p^{r+1}$ for all $r = 0, 1, \cdots, m-1$ in the diagram of L.

Case 2.1. $i \notin \{s : p_s^0 \neq p_s^m\}$. There exists $0 \leq q \leq m-1$ such that $s_i(p^q) \neq s_i(p^{q+1}) = s_i(x^1)$. By Lemma 3.5 and Lemma 3.6, there exists j such that $\widetilde{p^q}^j = p^{q+1}$. By lemma 5.4, $j \in \{s : p_s^q \neq p_s^{q+1}\} \subset \{s : p_s^0 \neq p_s^m\} = \{s : x_s^0 \neq x_s^1\} \subset I_F(x^0)$, and $p_i^q = p_i^{q+1}$. We may assume that

$s_i(p^{q+1}) = 1$. Thus $f_i(\widetilde{p^q}^j) = f_i(p^{q+1}) > p_i^{q+1} = p_i^q \geq f_i(p^q)$, it implies that $f_{ij}(p^q) = 1$. If $s_j(x^0) = 1$, then $f_j(x^0) > x_j^0 = 0_L$. Thus $x^0 < x^1$ and $p^q < p^{q+1}$, we have shown that $\Gamma(F'(p^q))$ has an edge from j to i with positive sign. If $s_j(x^0) = -1$, then $0_L = f_j(x^0) < x_j^0$. Thus $x^1 < x^0$ and $p^{q+1} < p^q$, we have shown that $\Gamma(F'(p^q))$ has an edge from j to i with negative sign. Thus $\Gamma(F'(p^q))$ has an edge from j to i with sign $s_j(x^0)s_i(x^k)$ when $i \notin \{s : p_s^0 \neq p_s^m\}$.

Case 2.2. $i \in \{s : p_s^0 \neq p_s^m\}$. By lemma 5.4, $i \in \{s : p_s^0 \neq p_s^m\} = \{s : x_s^0 \neq x_s^1\} \subset I_F(x^0)$, and $s_i(x^0)s_i(x^1) = s_i(p^0)s_i(p^m) = -1$. We may assume that $s_i(p^m) = 1$ and $s_i(p^0) = -1$. Since $f_i(p^m) > p_i^m$ and $f_i(p^0) < p_i^0$, it implies that $p_i^0 > p_i^m$ and $p^r > p^{r+1}$ for all $r = 0, 1, \cdots, m-1$ in the diagram of L. There exists $0 \leq q \leq m-1$ such that $f_i(p^q) = 0_L$ and $f_i(p^{q+1}) = a^i$. By Lemma 3.5 and Lemma 3.6, there exists j such that $\widetilde{p^q}^j = p^{q+1}$. It follows that $\Gamma(F'(p^q))$ has an edge from j to i with negative sign. We have shown that $\Gamma(F'(p^q))$ has an edge from j to i with sign $s_j(x^0)s_i(x^k)$ when $i \in \{s : p_s^0 \neq p_s^m\}$. This completes the proof of Case 2.

In case $k > 1$, let $i \in I_F(x^k)$. By the induction hypothesis, $\{x^{k-1}, x^k\}$ is a path of $G(F)$ of length 1.

Case 1. $d_p(x^{k-1}, x^k) = 1$. There exists $l \in I_F(x^{k-1})$ such that $x^k = \widetilde{x^{k-1}}^l$. Thus $\Gamma(F'(x^{k-1}))$ has an edge from l to i with sign $s_l(x^{k-1})s_i(x^k)$.

Case 2. $d_p(x^{k-1}, x^k) > 1$. There exists a path $\{x^{k-1} = p^0, p^1, \cdots, p^m = x^k\}$ which connects x^{k+1} and x^k with $p^r < p^{r+1}$ or $p^r > p^{r+1}$ for all $r = 0, 1, \cdots, m-1$ in the diagram of L.

Case 2.1. $i \notin \{s : p_s^0 \neq p_s^m\}$. Since $s_i(x^{k-1}) = s_i(p^0) \neq s_i(x^k) = s_i(p^m)$, $f_i(p^m) \neq p_i^m = p_i^0$. It implies that $p_i^0 = f_i(p^0)$. (If $p_i^0 \neq f_i(p^0)$, it implies that $s_i(p^0) = s_i(p^m)$, which is a contradiction.) There exists $0 \leq q \leq m-1$ such that $p_i^q = f_i(p^q)$ and $p_i^{q+1} \neq f_i(p^{q+1})$. By Lemma 3.5 and Lemma 3.6, there exists l such that $\widetilde{p^q}^l = p^{q+1}$. By lemma 5.4, $l \in \{s : p_s^q \neq p_s^{q+1}\} \subset \{s : p_s^0 \neq p_s^m\} = \{s : x_s^{k-1} \neq x_s^k\} \subset I_F(x^{k-1}) = I_F(p^0)$, and $s_i(x^k) = s_i(p^m) = s_i(p^{q+1})$. We may assume that $s_i(p^{q+1}) = 1$. Thus $f_i(\widetilde{p^q}^j) = f_i(p^{q+1}) > p_i^{q+1} = p_i^q = f_i(p^q)$, it implies that $f_{il}(p^q) = 1$. If $s_l(p^0) = 1$, then $f_l(p^0) > p_l^0 = 0_L$. It implies that $p^q < p^{q+1}$. Thus $\Gamma(F'(p^q))$ has an edge from l to i with positive sign. If $s_l(p^0) = -1$, then $0_L = f_l(p^0) < p_l^0$. It implies that $p^{q+1} < p^q$. Thus $\Gamma(F'(p^q))$ has an edge from l to i with negative sign. We have shown that $\Gamma(F'(p^q))$ has an edge from l to i with sign $s_l(x^{k-1})s_i(x^k)$ when $i \notin \{s : p_s^0 \neq p_s^m\}$.

Case 2.2. $i \in \{s : p_s^0 \neq p_s^m\}$. By lemma 5.4, $i \in \{s : p_s^0 \neq p_s^m\} = \{s : x_s^{k-1} \neq x_s^k\} \subset I_F(x^{k-1}) = I_F(p^0)$, and $s_i(x^{k-1})s_i(x^k) = s_i(p^0)s_i(p^m) = -1$. We may assume that $s_i(p^m) = 1$ and $s_i(p^0) = -1$. Since $f_i(p^m) > p_i^m$ and $f_i(p^0) < p_i^0$, it implies that $p_i^0 > p_i^m$ and $p^r > p^{r+1}$ for all $r = 0, 1, \cdots, m-1$ in the diagram of L. There exists $0 \leq q \leq m-1$ such that $f_i(p^q) = 0_L$ and $f_i(p^{q+1}) = a^i$. By Lemma 3.5 and Lemma 3.6, there exists l such that $\widetilde{p^q}^l = p^{q+1}$. It follows that $\Gamma(F'(p^q))$ has an edge from l to i with negative sign. We have shown that $\Gamma(F'(p^q))$ has an edge from l to i with sign $s_l(x^{k-1})s_i(x^k)$ when $i \in \{s : p_s^0 \neq p_s^m\}$. This completes the proof of Case 2.

According to Cases 1 and 2, there exist $l \in I_F(x^{k-1})$ and a point $x \in L$ such that $\Gamma(F'(x))$ has an edge from l to i with sign $s_l(x^{k-1})s_i(x^k)$. Consider the smallest $0 \leq p < k$ such that $s_l(x^p) = s_l(x^{k-1})$. If $p = 0$, then $l \in I_F(x^0)$ and $\Gamma(F'(x^{k-1}))$ has an edge from l to i with sign $s_l(x^0)s_i(x^k)$. Let $j = l$. Then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a path from j to i with sign $s_j(x^0)s_i(x^k)$. If $p > 0$, then $s_l(x^{i'}) \neq s_l(x^p)$ for all $0 \leq i' < p$. The path $\{x^0, \ldots, x^p\}$ satisfies the conditions of the lemma for $l \in I_F(x^p)$. By the induction hypothesis, there exists $j \in I_F(x^0)$ such that $\bigcup_{x \in L} \Gamma(F'(x))$ has a path from j to l with sign $s_j(x^0)s_l(x^p)$. Since $\Gamma(F'(x^{k-1}))$ has an edge from l to i with sign $s_l(x^{k-1})s_i(x^k)$, the graph $\bigcup_{x \in L} \Gamma(F'(x))$ contains a path from j to i with sign $s_j(x^0)s_l(x^p)s_l(x^{k-1})s_i(x^k)$. Since $s_l(x^p) = s_l(x^{k-1})$, $\bigcup_{x \in L} \Gamma(F'(x))$ contains a path from j to i with sign $s_j(x^0)s_i(x^k)$. ☐

Lemma 5.6 *L is a finite distributive lattice and $F : L \to L$. Let D be a cyclic attractor of $G(F)$. If there exists $x \in D$ such that $|I_F(x)| = 1$, then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit.*

Proof. Suppose that there exists $x^0 \in D$ such that $I_F(x) = \{i\}$. We have $d_p(x, \tilde{x}^i) = 1$ by Lemma 3.6. (If $d_p(x, \tilde{x}^i) \geq 2$, then there exists j such that $d_p(x, \tilde{x}^j) = 1$ and $x_j \neq \tilde{x}_j^i$. If $x_i = 0_L$, we have $x_j = 0_L$ and $a^j < a^i$. Since $f_i(x) = a^i$, it implies that $f_j(x) = a^j$ and $j \in I_F(x)$, which is a contradiction. Similarly, if $x_i = a^i$, then $x_j = a^j$ and $a^i < a^j$. Since $f_i(x) = 0_L$, it implies that $f_j(x) = 0_L$ and $j \in I_F(x)$, which is a contradiction.)

We need only to prove that the case $s_i(x^0) = 1$, the other case being similar. Let $x^1 = \widetilde{x^0}^i$. Then $G(F)$ has an edge from x^0 to x^1 and $x_i^0 < x_i^1$. Since $x^0 \in D$, we have $x^1 \in D$ and there exists a path $\{x^1, \ldots, x^k = x^0\}$ from x^1 to x^0, and $x^i \in D$ for all $i = 1, \ldots, k$. If $s_i(x^q) \geq 0$ for all $1 \leq q < k$, then $x_i^q \leq x_i^{q+1}$ for all $1 \leq q < k$. We have $x_i^0 < x_i^1 \leq x_i^k = x_i^0$, which is a contradiction. Thus, there exists a smallest $1 \leq p < k$ such that $f_i'(x^p) = -1$. We have $\{x_1, \ldots, x^p\}$ is a path in $G(F)$ and $i \in I_F(x^p)$, $s_i(x^q) \neq s_i(x^p)$ for all $0 \leq q < p$. According to Lemma 5.5, there exists $j \in I_F(x^0)$ such that $\bigcup_{x \in L} \Gamma(F'(x))$ has a path from j to i with sign $s_j(x^0)s_i(x^p)$. Since $I_F(x^0) = \{i\}$, we have $j = i$ and $s_j(x^0)s_i(x^p) = s_i(x^0)s_i(x^p) = -1$. Thus $\bigcup_{x \in L} \Gamma(F'(x))$ has a negative circuit from i to i. □

We proceed now to prove Theorem 5.1.

If there exists $x \in D$ such that $|I_F(x)| = 1$, then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit by Lemma 5.6. We may assume that $|I_F(x)| \geq 2$ for all $x \in D$. Let $J = \bigcup_{x \in D} I_F(x)$ and $i_1 \in J$ be such that $a^i \leq a^{i_1}$ or a^i and a^{i_1} are incomparable for all $i \in J$ with $i \neq i_1$. Let $H : L \to L$ be defined by

$$H(x) = \bigvee \{x_{i_1}, f_i(x) : i \neq i_1\}.$$

If $x \in D$, then $I_H(x) \subset I_F(x)$. Let $i \in I_H(x)$ and $h_i(x) \neq x_i$. If $i \notin I_F(x)$, then $f_i(x) = x_i$. If $x_i = a^i$, then $a^i \leq H(x)$ and $h_i(x) = a^i = x_i$, which is a contradiction. If $x_i = 0_L$, then $j \notin I_F(x)$ for all j such that $a^i \leq a^j$. Thus, $a^i \nleq H(x)$ and $h_i(x) = 0_L = x_i$, which is a contradiction. It follows that $I_H(x) \subset I_F(x)$.

We may assume that $x_{i_1} = 0_L$. If $i_i \notin I_F(x)$, then $H(x) = F(x)$ and hence $I_H(x) = I_F(x)$. If $i_i \in I_F(x)$, then $i_1 \notin H(x)$ by definition of H. If $i \in I_F(x) \setminus \{i_1\}$ and $x_i = 0_L$, then $f_i(x) = a^i$ and hence $h_i(x) = a^i$. We have $i \in I_H(x)$. If $i \in I_F(x) \setminus \{i_1\}$ and $x_i = a^i$, then $f_i(x) = 0_L$. Thus $f_j(x) = 0_L$ for all j such that $a^i \leq a^j$. We have $a^i \nleq H(x)$ and hence $h_i(x) = 0_L \neq x_i$ and $i \in I_H(x)$. In particular, if $x_{i_1} = 0_L$, then $I_H(x) = I_F(x) \setminus \{i_1\}$ when $i_i \in I_F(x)$ or $I_H(x) = I_F(x)$ when $i_i \notin I_F(x)$.

We now prove that D is a trap domain of $G(H)$. Let $x \in D$ and $I_H(x) \subset I_F(x)$. Since D is a trap domain of $G(F)$. If $i \in I_H(x) \subset I_F(x)$, then $\tilde{x}^i \in D$. It follows that D is a trap domain of $G(H)$. By definition, $G(H)$ contains at least one attractor $D' \subset D$. We may assume that $x_{i_1} = 0_L$ for all $x \in D'$. Then there exists $x \in D$ satisfies $x_{i_1} = 0_L$. Since $|I_F(x)| \geq 2$ and $I_H(x) = I_F(x) \setminus \{i_1\}$ when $i_i \in I_F(x)$ or $I_H(x) = I_F(x)$ when $i_i \notin I_F(x)$ for all $x \in D'$, we have $|D'| \geq 2$. It follows that D' is a cyclic attractor of $G(H)$. For $x \in D'$, consider the principle submatrix $M(x) = (m_{ij}(x))$ of $H'(x)$ defined by

$$m_{ij}(x) = \begin{cases} h_{ij}(x) & i, j \in J, \\ 0 & \text{otherwise.} \end{cases}$$

Since $x_{i_1} = 0_L$, we have $h_i(x) = f_i(x)$ for all $i \in J$ and $M(x)$ is the principle submatrix of $F'(x)$. We have $\Gamma(M(x))$ is a subgraph of $\Gamma(F'(x))$. If there exists $x \in D'$ such that $|I_H(x)| = 1$, then $\bigcup_{x \in L} \Gamma(M(x))$ has a negative circuit by Lemma 5.6. Thus $\bigcup_{x \in L} \Gamma(F'(x))$ has a negative circuit.

If $|I_H(x)| \geq 2$ for all $x \in D'$, consider $J' = \bigcup_{x \in L} I_H(x)$ and $i_2 \in J'$ such that $a^i \leq a^{i_2}$ or a^i and a^{i_2} are incomparable for all $i \in J' \setminus \{i_2\}$. Repeating the above process, we obtain a function $\bar{H}$ and a cyclic attractor $\bar{D}$ such that $\bar{D} \subset D$ and $|I_{\bar{H}}(x)| = 1$ for some $x \in \bar{D}$. It follows that $\bigcup_{x \in L} \Gamma(F'(x))$ has a negative circuit and the proof is completed.

6 Concluding remarks

The distributive condition in Theorem 4.3 is required. We have the following example.

Example 6.1 *Let L be the lattice with 16 elements. The diagram of L is shown in figure 2 which $\mathcal{J}(L) = \{a^i; i = 1, \cdots, 11\}$. Because*

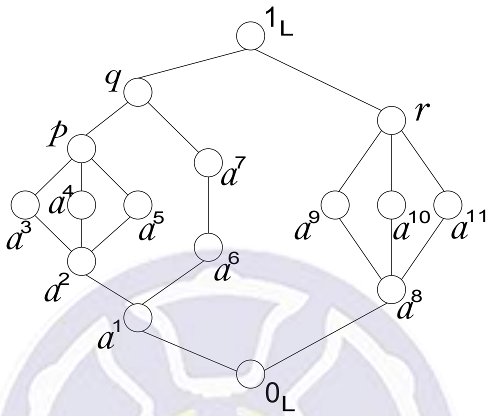

Figure 2: The diagram of L

$$a^9 \wedge (a^{10} \vee a^{11}) = a^9 \wedge r = a^9 \neq a^8 = a^8 \vee a^8 = (a^9 \wedge a^{10}) \vee (a^9 \wedge a^{11}),$$

L does not satisfy the distributive law. Consider the map $F : L \to L$ is the following:

x	0_L	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	a^{11}	p	q	r	1_L
$F(x)$	1_L	1_L	a^2	a^2	a^2	a^2	1_L	a^6	a^8	a^8	a^8	a^8	a^6	a^6	a^6	a^6

The generalized Boolean Jacobian matrix of F are the following:

$$F'(0) = F'(a^8) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^1) = F'(a^2) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^3) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^4) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^6) = F'(a^7) \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^9) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^{10}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$F'(a^{11}) = \begin{pmatrix} 0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&1&1&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix},$$

$$F'(p) = F'(q) = F'(r) = F'(1_L) = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix}.$$

Therefore F has 2 fixed points and $\rho(F'(x)) = 0$ for all $x \notin \{a^6, a^7\}$, thus the interaction graph $\Gamma(F'(x))$ has no circuit. The interaction graph of $\Gamma(F'(a^6)) = \Gamma(F'(a^7))$ is the following directed graph which contains a negative circuit.

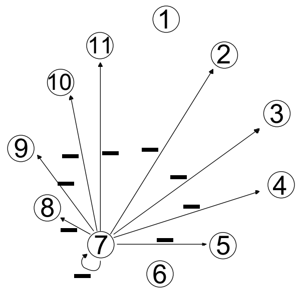

Figure 3: The interaction graph of $\Gamma(F'(a^6))$ and $\Gamma(F'(a^7))$

7 Related open questions

Theorem 3.3 may be extended to finite lattice from finite distributive lattice. It is stated as follows.

Conjecture 7.1 *Let L be a finite lattice with $\mathcal{J}(L) = \{a^1, \ldots, a^k\}$ and let $F : L \to L$ be such that $\Gamma(F'(x))$ has no circuit for all $x \in L$. Then for every $a, b \in L$ with $a < b$ and $I = \{i : a_i < b_i\}$, there exists a unique point $\alpha \in [a, b]$ such that $(F(\alpha))_i = \alpha_i$ for all $i \in I$.*

The uniqueness assertion of Conjecture 7.1 is easy to prove by induction on $d_p(a, b)$. However, the existence assertion is difficult to prove. Thus Conjecture 7.1 remains open.

Conjecture 7.2 *Let L be a finite distributive lattice. If $F : L \to L$ has no fixed point, then there exists a point $x \in L$ such that $\Gamma(F'(x))$ has a negative circuit.*

Conjecture 7.2 comes from the equivalent contrapositive form of Theorem 3.1. The equivalent contrapositive form of Theorem 3.1 is stated as follows. Let L be a finite distributive lattice. If $F : L \to L$ has multiple fixed points or has no fixed point, then there exists a point $x \in \{0, 1\}^n$ such that the corresponding network $\Gamma(F'(x))$ has a circuit.
Theorem 4.3 shows that if F has multiple fixed points, then there exists a point $x \in \{0, 1\}^n$ such that the corresponding network $\Gamma(F'(x))$ has a circuit. On the other hand, Theorem 5.2 states that if F has no fixed point, then $\bigcup_{x \in L} \Gamma(F'(x))$ contains a negative circuit. However, if F has no fixed

point, then there exists a point $x \in L$ such that $\Gamma(F'(x))$ has a circuit. In view of Thomas' conjecture, the sign of the circuit in $\Gamma(F'(x))$ should be negative.

References

[1] H. Bass, E. Connell, and D. Wright, The Jacobian conjecture: reduction on degree and formal expansion of the inverse., Bull. Amer. Math. Soc., **7** No.2 (1982), 287–330.

[2] G. Birkhoff, Rings of sets, Duke Mathematical Journal, **3** (1937), 443–454.

[3] G. Birkhoff, Lattice Theory, Third ed., Amer. Math. Soc., Providence, Rhode Island, 1967.

[4] A. Cima, A. Gasull, and F. Mañosas, The discrete Markus-Yamabe problem, Nonlinear analysis, **35** (1999), 343–354.

[5] O. H. Keller, Ganze Cremona-Transformationen, Monatshefte für Mathematik und Physik, **47** Issue 1 (1939), 299–306.

[6] E. Remy, P. Ruet, and D. Thieffry, Graphic requirements for multistability and attractive cycles in a Boolean dynamical framework, Advances in Applied mathematics, **41** (2008), 335–350.

[7] A. Richard and J.-P. Comet, Necessary conditions for multistationarity in discrete dynamical systems, Discrete Applied Mathematics, **155** (2007), 2403–2413.

[8] A. Richard, An extension of a combinatorial fixed point theorem of Shih and Dong, Advances in Applied mathematics, **41** (2008), 620–627.

[9] A. Richard, Positive circuits and maximal number of fixed points in discrete dynamical systems, Discrete Applied Mathematics, **157** (2009), 3281–3288.

[10] F. Robert, Discrete Iteration, A Metric Study, Springer Ser. Comput. Math., vol. 6, Springer-Verlag, Berlin-Heidelberg-New York, 1986.

[11] W. Rudin, Injective polynomial maps are automorphisms, Amer. Math. Monthly, **102** No.6 (1995), 540–543.

[12] M.-H. Shih and J.-L. Ho, Solution of the Boolean MarkusVYamabe problem, Advances in Applied Mathematics, **22** (1999), 60–102.

[13] M.-H. Shih and J.-L. Dong, A combinatorial analogue of the Jacobian problem in automata networks, Advances in Applied Mathematics, **34** (2005), 30–46.

[14] S. Smale, Mathematical problems for the next century, Mathematical Intelligencer, **20** No. 2 (1998), 7–15.

[15] C. Soulé, Mathematical approaches to differentiation and gene regulation, C.R. Paris Biologies, **329** (2006), 13–20.

[16] R. Thomas, On the relation between the logical structure of systems and their ability to generate multiple steady states or sustained oscillations, Springer Series in Synergetics, **9** (1981), 180–193.

[17] R. Thomas and R. d'Ari, Biological Feedback, CRC Press, 1990.

[18] R. Thomas, Laws for the dynamics of regulatory networks, Internat. J. Dev. Biol., **42** (1998), 479–485.

[19] R. Thomas and M. Kaufman, Multistationarity, the basis of cell differentiation and memory. I. and II., Chaos, **11** (2001) 170-195.