1 Introduction

In this paper, we discuss the positive integer solutions of the Diophantine equations $x^2 + y^2 + z^2 = kxyz$, where x, y, z, $k \in \mathbf{N}$. Hurwitz [6] showed that $x^2 + y^2 + z^2 = kxyz$ has positive integer solutions only when $k = 1$ or $k = 3$. In Section 2 and Section 3 of this paper, we give simple proofs to illustrate that it has no positive integer solutions when $k \geq 4$ and $k = 2$.

When $k = 1$, we can easily observe that the equation has many solutions. Not to distinguish the solution from others obtained by permuting its entries, we usually arrange its entries in ascending order. In ascending order of the largest entry of them, the first 14 solutions are

$$(3,3,3),(3,3,6),(3,6,15),(3,15,39),(6,15,87),(3,39,102),(3,102,267),(6,87,507),$$
$$(15,39,582),(3,267,699),(15,87,1299),(3,699,1830),(6,507,2955),(39,102,3975).$$

We say that the two solutions $(3,3,3)$ and $(3,3,6)$ are singular, and the others are nonsingular. It is easy to show that the entries of a nonsingular solution are all distinct. Next, we will show that the equation $x^2 + y^2 + z^2 = xyz$ has infinitely many solutions. (See Remark 4.2 (1).)

The 14 solutions above give us an impression that the largest entries from each solution are all distinct. Thus, we have the following conjecture:

Unicity Conjecture. If (x, y, z) and (x_1, y_1, z) are solutions, where $x \leq y \leq z$ and $x_1 \leq y_1 \leq z$, then $x = x_1$ and $y = y_1$.

We only can prove the conjecture for some special cases.

Theorem 1.1 *A solution (a, b, c) is determined uniquely by c if c is 3 times a prime power or 6 times a prime power.*

Next, we can easily prove the following Lemma.

Lemma 4.11 *Let (a, b, c) be a solution of $x^2 + y^2 + z^2 = xyz$. Then*
(i) $gcd(a, b) = gcd(b, c) = gcd(c, a) = 3$
(ii) *Every odd entry* $\equiv 3 \pmod 4$
(iii) *Every even entry* $\equiv 6 \pmod 8$

Further, we can obtain the following simple but detailed congruence for all even entries of the solutions.

Lemma 4.13 *If c is an even entry then* $c \equiv 6 \pmod{32}$.

This situation is the most possible since the first two even entries are 6 and 102. As a consequence of Lemma 4.13, we see that for an even entry c,

$$c - 2 \equiv 4 \pmod{32} \qquad \text{and} \qquad c + 2 \equiv 8 \pmod{32}.$$

Hence $c - 2$ is 4 times an odd number and $c + 2$ is 8 times an odd number. So we have the following Theorem.

Theorem 1.2 *A solution (a, b, c) is determined uniquely by c if c satisfies:*
(i) $c + 2$ or $c - 2$ is a prime power when c is odd, or
(ii) $c - 2$ is 4 times a prime power or $c + 2$ is 8 times a prime power when c is even.

In Section 4, we will give proofs of the above lemmas and theorems.

When $k = 3$, the equation $x^2 + y^2 + z^2 = 3xyz$ is the well known Markoff equation. It has infinitely many positive integer solutions. We say that the positive integer solutions are Markoff triples. The following conjecture was made by G. Frobenius [5] in 1913.

Markoff Conjecture. If (a, b, c) and (a_1, b_1, c) are markoff triples, where $a \leq b \leq c$ and $a_1 \leq b_1 \leq c$, then $a = a_1$ and $b = b_1$.

Up to now, the Markoff Conjecture is still open. In Section 5, we list some results.