第三章　純量粒子在 Schwarzschild 背景中

3.1　方程式的推導

靜止質量為m_0的純量粒子在質量為M的 Schwarzschild 黑洞背景中，粒子所遵循的方程式為

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\partial^\mu\Phi\right)-m_0{}^2\Phi=0 \qquad (3.1)$$

其中$\partial_\mu=\dfrac{\partial}{\partial x^\mu}$，$\partial^\mu=g^{\mu\nu}\partial_\nu$，由 Schwarzschild 度規可得$g=\det\left(g_{\mu\nu}\right)=-r^4\sin^2\theta$，代入(3.1)式

$$\frac{1}{r^2\sin\theta}\partial_\mu\left(r^2\sin\theta\,\partial^\mu\Phi\right)-m_0{}^2\Phi=0$$

$$\frac{1}{r^2\sin\theta}\left\{\begin{aligned}&\frac{\partial}{\partial t}\left[-r^2\sin\theta\frac{1}{(1-2M/r)}\frac{\partial}{\partial t}\Phi\right]+\frac{\partial}{\partial r}\left[r^2\sin\theta(1-2M/r)\frac{\partial}{\partial r}\Phi\right]\\&+\frac{\partial}{\partial\theta}\left[r^2\sin\theta\frac{1}{r^2}\frac{\partial}{\partial\theta}\Phi\right]+\frac{\partial}{\partial\varphi}\left[r^2\sin\theta\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\Phi\right]\end{aligned}\right\}-m_0{}^2\Phi=0$$

整理可得

$$\left[\begin{aligned}&-\frac{1}{(1-2M/r)}\frac{\partial^2\Phi}{\partial t^2}+(1-2M/r)\frac{\partial^2\Phi}{\partial r^2}\\&+\frac{1}{r^2}\left\{\frac{\partial}{\partial r}\left[r^2(1-2M/r)\right]\right\}\frac{\partial\Phi}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right)\\&+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\varphi^2}-m_0{}^2\Phi\end{aligned}\right]=0 \qquad (3.2)$$

令波函數

$$\Phi=\exp(-i\omega t)f(r)Y_{lm}(\theta,\varphi) \qquad (3.3)$$

將(3.3)式代入(3.2)式，並且利用(2.9)式與(2.11)式，(3.2)式可變成

$$\left[\begin{aligned}&\left[r^2(1-2M/r)\right]\frac{d^2f}{dr^2}+\left[2r(1-M/r)\right]\frac{df}{dr}\\&+\left[\frac{r^2}{(1-2M/r)}\omega^2-l(l+1)-m_0{}^2r^2\right]f(r)\end{aligned}\right]=0 \qquad (3.4)$$

將(3.4)式以完整單位的形式寫出

$$\left[\begin{array}{l}\left[r^2\left(1-\frac{2GM/c^2}{r}\right)\right]\frac{d^2f}{dr^2}+\left[2r\left(1-\frac{GM/c^2}{r}\right)\right]\frac{df}{dr}\\+\left[\frac{r^2}{\left(1-\frac{2GM/c^2}{r}\right)}\left(\frac{E}{\hbar c}\right)^2-l(l+1)-\left(\frac{m_0c}{\hbar}\right)^2r^2\right]f(r)\end{array}\right]=0 \qquad (3.5)$$

令

$$x(r)=r+\left(2GM/c^2\right)\ln\left[\frac{r}{\left(2GM/c^2\right)}-1\right]$$

$$f=\frac{\phi}{r} \qquad (3.6)$$

(3.5)式變成

$$\frac{d^2\phi}{dx^2}+\left\{\left(\frac{E}{\hbar c}\right)^2-\left[\begin{array}{l}\left(\frac{2GM}{c^2}\right)\frac{\left(1-\frac{2GM/c^2}{r}\right)}{r^3}\\+l(l+1)\frac{\left(1-\frac{2GM/c^2}{r}\right)}{r^2}+\left(\frac{m_0c}{\hbar}\right)^2\left(1-\frac{2GM/c^2}{r}\right)\end{array}\right]\right\}\phi=0 \qquad (3.7)$$

(3.7)式可再進一步化成

$$\frac{d^2\phi}{dx^2}+\left\{\left[\left(\frac{E}{\hbar c}\right)^2-\left(\frac{m_0c}{\hbar}\right)^2\right]-V\right\}\phi=0 \qquad (3.8)$$

其中位能

$$V=\left(1-\frac{2GM/c^2}{r}\right)\left[\frac{l(l+1)}{r^2}+\frac{2GM/c^2}{r^3}\right]-\left(\frac{m_0c}{\hbar}\right)^2\left(\frac{2GM/c^2}{r}\right)$$

當 $m_0=0$ 時，(3.8)式與(2.34)式的形式相同，只有位能不同而已，這兩個方程式的位能可以由下列形式的位能給出

$$V=\left(1-\frac{2GM/c^2}{r}\right)\left[\frac{l(l+1)}{r^2}+\sigma\frac{2GM/c^2}{r^3}\right]$$

$$\sigma=\begin{cases}+1 & \text{scalar test field}\\-3 & \text{axial gravitational perturbation}\end{cases}$$

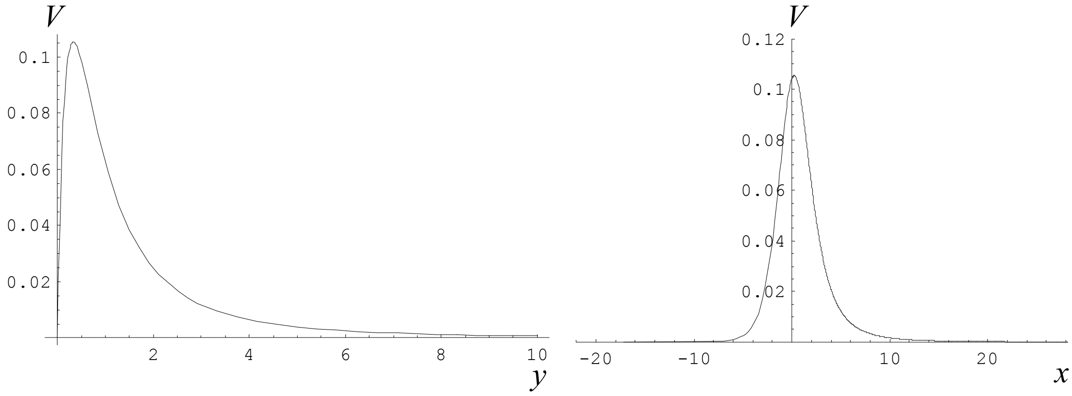
$\sigma=1-\tau^2$，其中 $\tau\in\{0,2\}$ 是微擾場的自旋。

令 $y=\dfrac{r}{2GM/c^2}-1$，位能變成

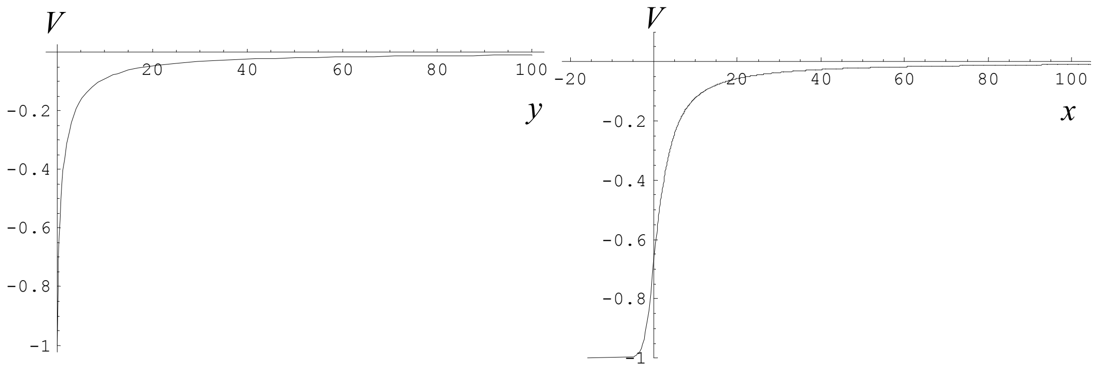
$$V=\frac{1}{\left(2GM/c^2\right)^2}\left[\frac{y}{(y+1)^4}+l(l+1)\frac{y}{(y+1)^3}\right]-\left(\frac{m_0c}{\hbar}\right)^2\frac{1}{y+1}$$

相對論自由粒子的能量為 $E^2=c^2\vec{p}^2+m_0^{\ 2}c^4$，比較(3.8)式，可知靜止質量為 m_0 的純量粒子在重力場中運動受到一等效位能 V，這與帶電荷 q 的粒子在電磁場中運動的動量算符 $\vec{p}=-i\hbar\nabla-\dfrac{q}{c}\vec{A}$ 不盡相同。

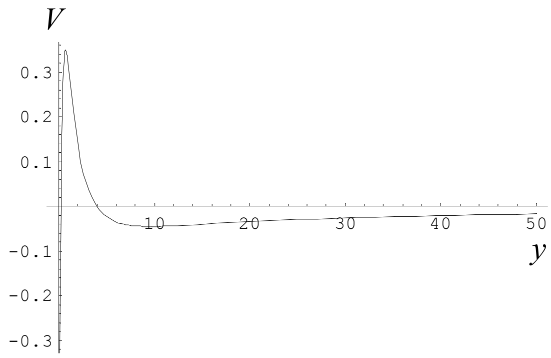
$m_0=0$，$l=0$，$2GM/c^2=1$ 的位能圖



$m_0c/\hbar=1$，$l=0$，$2GM/c^2=1$ 的位能圖



$m_0 c/\hbar = 1$，$l = 2$，$2GM/c^2 = 1$ 的位能圖



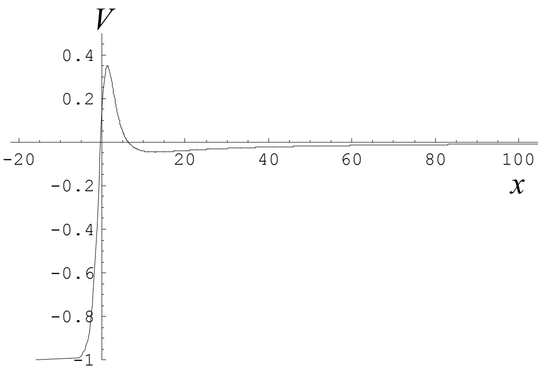


圖 3.1：在以上各圖中，左圖 horizon 位在 $y = 0$，右圖 horizon 位在 $x = -\infty$。

3.2 連續態

當 $x \to -\infty$（即 $y \to 0$ 或 $r \to 2GM/c^2$）時，$V \to -(m_0 c/\hbar)^2$，(3.8)式變成

$$\frac{d^2\phi}{dx^2} + \left(\frac{E}{\hbar c}\right)^2 \phi = 0 \qquad (3.9)$$

已知 $x = \frac{2GM}{c^2}\left[y + 1 + \ln(y)\right]$，由(3.9)式解得

$$\begin{aligned}\phi &= a_0{''}\exp\left[i\left(\frac{E}{\hbar c}\right)x\right] + b_0{''}\exp\left[-i\left(\frac{E}{\hbar c}\right)x\right] \\ &= a_0{'}\exp\{i2\zeta[y + \ln(y)]\} + b_0{'}\exp\{-i2\zeta[y + \ln(y)]\}\end{aligned}$$

再從(3.6)式可知

$$f(y) = a_0\frac{\exp\{i2\zeta[y + \ln(y)]\}}{y+1} + b_0\frac{\exp\{-i2\zeta[y + \ln(y)]\}}{y+1} \qquad (3.10)$$

(3.10)式是當 $r \to 2GM/c^2$ 時的徑向波函數。

當 $x \to \infty$（即 $y \to \infty$ 或 $r \to \infty$）時，$V \to 0$，(3.8)式變成

$$\frac{d^2\phi}{dx^2} + \left[\left(\frac{E}{\hbar c}\right)^2 - \left(\frac{m_0 c}{\hbar}\right)^2\right]\phi = 0 \qquad (3.11)$$

(3.11)式如預期回到自由粒子的 Klein-Gordon 方程式，可解得

$$f(y) = C_1\frac{\exp\left\{i2\zeta\sqrt{1-(\eta/\zeta)^2}\,[y + \ln(y)]\right\}}{y+1} + C_2\frac{\exp\left\{-i2\zeta\sqrt{1-(\eta/\zeta)^2}\,[y + \ln(y)]\right\}}{y+1} \quad (3.12)$$

其中$\eta=\dfrac{GMm_0}{\hbar c}$，(3.12)式是當$r\to\infty$時的徑向波函數。當(3.12)式中的$\zeta>\eta$時，即$E>m_0c^2$，為一般的散射態。

若$\zeta<\eta$，即$0<E<m_0c^2$，(3.12)式變成

$$f(y)=C_1\frac{\exp\left\{-2\sqrt{\eta^2-\zeta^2}\left[y+\ln(y)\right]\right\}}{y+1}+C_2\frac{\exp\left\{2\sqrt{\eta^2-\zeta^2}\left[y+\ln(y)\right]\right\}}{y+1}$$

當$x\to\infty$（即$y\to\infty$或$r\to\infty$）時，我們要求波函數必須為有限值，故取$C_2=0$，上式變成

$$f(y)=C_1\frac{\exp\left\{-2\sqrt{\eta^2-\zeta^2}\left[y+\ln(y)\right]\right\}}{y+1} \qquad (3.13)$$

但是當$x\to-\infty$（即$y\to 0$或$r\to 2GM/c^2$）時徑向波函數是振盪的形式，因此整體而言波函數還是屬於連續態，這和我們原本所預期的不一樣。

3.3 連續性方程式

將(3.2)式以完整單位的形式寫出

$$\left[\begin{array}{l}\left[-\dfrac{1}{\left(1-\dfrac{2GM/c^2}{r}\right)}\dfrac{1}{c^2}\right]\dfrac{\partial^2\Phi}{\partial t^2}+\left(1-\dfrac{2GM/c^2}{r}\right)\dfrac{\partial^2\Phi}{\partial r^2}\\ +\dfrac{1}{r^2}\left\{\dfrac{\partial}{\partial r}\left[r^2\left(1-\dfrac{2GM/c^2}{r}\right)\right]\right\}\dfrac{\partial\Phi}{\partial r}+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial\Phi}{\partial\theta}\right)\\ +\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2\Phi}{\partial\varphi^2}-\left(\dfrac{m_0c}{\hbar}\right)^2\Phi\end{array}\right]=0 \qquad (3.14)$$

將(3.3)式代入(3.14)式，並且利用(2.9)式與(2.11)式，(3.14)式可變成

$$\left[\begin{array}{l}\left[-\dfrac{1}{\left(1-\dfrac{2GM/c^2}{r}\right)}\dfrac{1}{c^2}\right]\dfrac{\partial^2\Phi}{\partial t^2}+\left(1-\dfrac{2GM/c^2}{r}\right)\dfrac{\partial^2\Phi}{\partial r^2}\\ +\left[\dfrac{2}{r}\left(1-\dfrac{GM/c^2}{r}\right)\right]\dfrac{\partial\Phi}{\partial r}+\left[-\dfrac{l(l+1)}{r^2}-\left(\dfrac{m_0c}{\hbar}\right)^2\right]\Phi\end{array}\right]=0$$

令 $y=\dfrac{r}{2GM/c^2}-1$，上式再變成

$$\left[\begin{array}{l}\left[-\left(\dfrac{2GM}{c^2}\right)^2\dfrac{(y+1)^3}{y}\dfrac{1}{c^2}\right]\dfrac{\partial^2\Phi}{\partial t^2}+[y(y+1)]\dfrac{\partial^2\Phi}{\partial y^2}+(2y+1)\dfrac{\partial\Phi}{\partial y}\\ +\left[-l(l+1)-\left(\dfrac{m_0c}{\hbar}\right)^2\left(\dfrac{2GM}{c^2}\right)^2(y+1)^2\right]\Phi\end{array}\right]=0 \tag{3.15}$$

$\Phi^*\times(3.15)-\Phi\times(3.15)^*$，可得出

$$\left[\begin{array}{l}\left[-\left(\dfrac{2GM}{c^2}\right)^2\dfrac{(y+1)^3}{y}\dfrac{1}{c^2}\right]\left(\Phi^*\dfrac{\partial^2\Phi}{\partial t^2}-\Phi\dfrac{\partial^2\Phi^*}{\partial t^2}\right)\\ +[y(y+1)]\left(\Phi^*\dfrac{\partial^2\Phi}{\partial y^2}-\Phi\dfrac{\partial^2\Phi^*}{\partial y^2}\right)+(2y+1)\left(\Phi^*\dfrac{\partial\Phi}{\partial y}-\Phi\dfrac{\partial\Phi^*}{\partial y}\right)\end{array}\right]=0$$

$$\frac{\partial}{\partial t}\left[-\left(\frac{2GM}{c^2}\right)^2\frac{(y+1)^3}{y}\frac{1}{c^2}\left(\Phi^*\frac{\partial\Phi}{\partial t}-\Phi\frac{\partial\Phi^*}{\partial t}\right)\right]+\frac{\partial}{\partial y}\left[y(y+1)\left(\Phi^*\frac{\partial\Phi}{\partial y}-\Phi\frac{\partial\Phi^*}{\partial y}\right)\right]=0$$

$$\frac{\partial}{\partial t}\left[-\frac{(y+1)^3}{y}\frac{1}{c^2}\left(\Phi^*\frac{\partial\Phi}{\partial t}-\Phi\frac{\partial\Phi^*}{\partial t}\right)\right]+\frac{\partial}{\partial r}\left[y(y+1)\left(\Phi^*\frac{\partial\Phi}{\partial r}-\Phi\frac{\partial\Phi^*}{\partial r}\right)\right]=0$$

上式與連續性方程式 $\dfrac{\partial}{\partial t}\rho+\dfrac{\partial}{\partial r}j=0$ 比較可得機率流密度

$$j\propto y(y+1)\left(\Phi^*\frac{\partial\Phi}{\partial r}-\Phi\frac{\partial\Phi^*}{\partial r}\right) \tag{3.16}$$

3.4 機率守恆關係式

現在考慮一物理情況，即純量粒子從距離黑洞無限遠處往黑洞視界正向入射，求其反射率與透射率。假設在距離黑洞無限遠處的入射徑向波函數及反射徑向波函數為(3.12)式所示，透射徑向波函數為(3.10)式的第二項。

入射徑向波函數： $f_i(y) = C_2 \dfrac{\exp\left\{-i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{y+1}$

反射波徑向函數： $f_r(y) = C_1 \dfrac{\exp\left\{i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{y+1}$

透射徑向波函數： $f_t(y) = b_0 \dfrac{\exp\{-i2\zeta[y+\ln(y)]\}}{y+1}$

由於波函數 $\Phi = \exp(-i\omega t) f(y) Y_{lm}(\theta,\varphi)$ ，可知入射波函數

$$\Phi_i = \exp(-i\omega t) C_2 \frac{\exp\left\{-i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{y+1} Y_{lm}(\theta,\varphi)$$

$$\Phi_i^* = \exp(i\omega t) C_2^* \frac{\exp\left\{i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{y+1} Y_{lm}^*(\theta,\varphi)$$

$$\begin{aligned}\frac{\partial\Phi_i}{\partial r} &= \left(\frac{\partial\Phi_i}{\partial y}\right)\left(\frac{\partial y}{\partial r}\right) \\ &= \frac{\exp(-i\omega t) C_2 Y_{lm}(\theta,\varphi)}{\left(2GM/c^2\right)} \\ &\times \frac{\left[-i2\zeta\sqrt{1-(\eta/\zeta)^2}(1+1/y)(y+1)-1\right]\exp\left\{-i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{(y+1)^2}\end{aligned}$$

由(3.16)式可知入射機率流密度

$$j_i \propto y(y+1)\left(\Phi_i^* \frac{\partial\Phi_i}{\partial r} - \Phi_i \frac{\partial\Phi_i^*}{\partial r}\right) = \frac{|C_2|^2 |Y_{lm}(\theta,\varphi)|^2 \left[-i4\zeta\sqrt{1-(\eta/\zeta)^2}\right]}{\left(2GM/c^2\right)}$$

已知反射波函數

$$\Phi_r = \exp(-i\omega t) C_1 \frac{\exp\left\{i2\zeta\sqrt{1-(\eta/\zeta)^2}\left[y+\ln(y)\right]\right\}}{y+1} Y_{lm}(\theta,\varphi)$$

由(3.16)式可知反射機率流密度

$$j_r \propto y(y+1)\left(\Phi_r^{\ *}\frac{\partial \Phi_r}{\partial r}-\Phi_r\frac{\partial \Phi_r^{\ *}}{\partial r}\right)=\frac{|C_1|^2|Y_{lm}(\theta,\varphi)|^2\left[i4\zeta\sqrt{1-(\eta/\zeta)^2}\right]}{(2GM/c^2)}$$

已知透射波函數

$$\Phi_t=\exp(-i\omega t)b_0\frac{\exp\{-i2\zeta[y+\ln(y)]\}}{y+1}Y_{lm}(\theta,\varphi)$$

由(3.16)式可知透射機率流密度

$$j_t \propto y(y+1)\left(\Phi_t^{\ *}\frac{\partial \Phi_t}{\partial r}-\Phi_t\frac{\partial \Phi_t^{\ *}}{\partial r}\right)=\frac{|b_0|^2|Y_{lm}(\theta,\varphi)|^2(-i4\zeta)}{(2GM/c^2)}$$

反射率

$$R\equiv\left|\frac{j_r}{j_i}\right|=\left|\frac{\dfrac{|C_1|^2|Y_{lm}(\theta,\varphi)|^2\left[i4\zeta\sqrt{1-(\eta/\zeta)^2}\right]}{(2GM/c^2)}}{\dfrac{|C_2|^2|Y_{lm}(\theta,\varphi)|^2\left[-i4\zeta\sqrt{1-(\eta/\zeta)^2}\right]}{(2GM/c^2)}}\right|=\frac{|C_1|^2}{|C_2|^2} \tag{3.17}$$

透射率

$$T\equiv\left|\frac{j_t}{j_i}\right|=\left|\frac{\dfrac{|b_0|^2|Y_{lm}(\theta,\varphi)|^2(-i4\zeta)}{(2GM/c^2)}}{\dfrac{|C_2|^2|Y_{lm}(\theta,\varphi)|^2\left[-i4\zeta\sqrt{1-(\eta/\zeta)^2}\right]}{(2GM/c^2)}}\right|=\frac{1}{\sqrt{1-(\eta/\zeta)^2}}\frac{|b_0|^2}{|C_2|^2} \tag{3.18}$$

我們可經由以下的過程確認機率守恆關係式 $R+T=1$。(3.7)式可變成

$$y(y+1)\frac{d^2 f}{dy^2}+(2y+1)\frac{df}{dy}+\left[4\zeta^2\frac{(y+1)^3}{y}-l(l+1)-4\eta^2(y+1)^2\right]f(y)=0 \tag{3.19}$$

$f^*(y)\times(3.19)-f(y)\times(3.19)^*$，可得出

$$y(y+1)\left[f^*(y)\frac{d^2 f}{dy^2}-f(y)\frac{d^2 f^*}{dy^2}\right]+(2y+1)\left[f^*(y)\frac{df}{dy}-f(y)\frac{df^*}{dy}\right]=0$$

$$\Rightarrow \frac{d}{dy}\left\{y(y+1)\left[f^*(y)\frac{df}{dy}-f(y)\frac{df^*}{dy}\right]\right\}=0$$

從上式可得一與 y 無關的式子

$$y(y+1)\left[f^*(y)\frac{df}{dy}-f(y)\frac{df^*}{dy}\right]=const. \qquad \forall y \tag{3.20}$$

當 $y\to\infty$ 時，徑向波函數

$$f(y)=C_1\frac{\exp\left\{i2\zeta\sqrt{1-(\eta/\zeta)^2}\,[y+\ln(y)]\right\}}{y+1}+C_2\frac{\exp\left\{-i2\zeta\sqrt{1-(\eta/\zeta)^2}\,[y+\ln(y)]\right\}}{y+1}$$

可得

$$y(y+1)\left[f^*(y)\frac{df(y)}{dy}-f(y)\frac{df^*(y)}{dy}\right]=i4\zeta\sqrt{1-(\eta/\zeta)^2}\left(|C_1|^2-|C_2|^2\right)$$

當 $y\to 0$ 時，徑向波函數

$$f(y)=b_0\frac{\exp\{-i2\zeta[y+\ln(y)]\}}{y+1}$$

可得

$$y(y+1)\left[f^*(y)\frac{df(y)}{dy}-f(y)\frac{df^*(y)}{dy}\right]=-i4\zeta|b_0|^2$$

由(3.20)式可知

$$i4\zeta\sqrt{1-(\eta/\zeta)^2}\left(|C_1|^2-|C_2|^2\right)=-i4\zeta|b_0|^2$$
$$\Rightarrow\frac{|C_1|^2}{|C_2|^2}+\frac{1}{\sqrt{1-(\eta/\zeta)^2}}\frac{|b_0|^2}{|C_2|^2}=1$$

故得機率守恆關係式 $R+T=1$ 。

3.5　反射率與透射率的 WKB 近似法

將(3.7)式寫成

$$\frac{d^2\phi}{dx^2}+\left[\left(\frac{E}{\hbar c}\right)^2-V\right]\phi=0 \qquad (3.21)$$

其中

$$V=\left(\frac{2GM}{c^2}\right)\frac{\left(1-\frac{2GM/c^2}{r}\right)}{r^3}+l(l+1)\frac{\left(1-\frac{2GM/c^2}{r}\right)}{r^2}+\left(\frac{m_0c}{\hbar}\right)^2\left(1-\frac{2GM/c^2}{r}\right)$$
$$=\frac{1}{(2GM/c^2)^2}\left[\frac{y}{(y+1)^4}+l(l+1)\frac{y}{(y+1)^3}\right]+\left(\frac{m_0c}{\hbar}\right)^2\frac{y}{y+1}$$

令 $Q^2=\left(\frac{E}{\hbar c}\right)^2-V$ ，代入(3.21)式

$$\frac{d^2\phi}{dx^2}+Q^2\phi=0 \qquad (3.22)$$

(3.22)式的 WKB 近似法的解

$$\phi_{1,2}^{(t)}(x)=Q^{-1/2}(x)\exp\left[\pm i\int_t^x Q(x')dx'\right]$$

$m_0c/\hbar=1$，$l=3$，$2GM/c^2=1$的V 圖

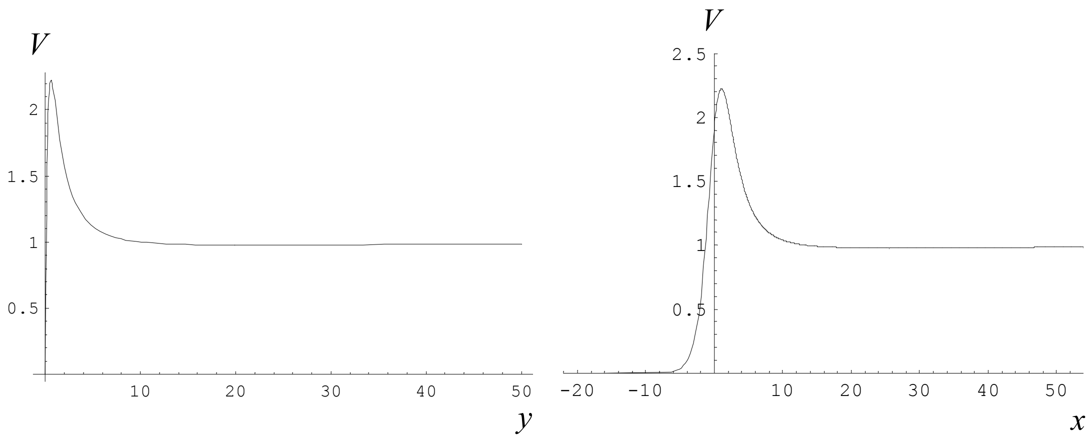


圖 3.2：左圖 horizon 位在$y=0$，右圖 horizon 位在$x=-\infty$。

現在考慮的物理情況與 3.4 節所述物理情況相同，即純量粒子從距離黑洞無限遠處往黑洞視界正向入射，求其反射率與透射率。Region Ⅰ為距離黑洞無限遠處至$(E/\hbar c)^2=V$ 處的區域，Region Ⅱ為$(E/\hbar c)^2<V$ 的區域，Region Ⅲ為$(E/\hbar c)^2=V$ 處至黑洞視界處的區域，則

Region Ⅰ： $\phi_{\mathrm{I}}(x)=Q^{-1/2}(x)\exp\left[-i\int_{x_2}^x Q(x')dx'\right]+CQ^{-1/2}(x)\exp\left[i\int_{x_2}^x Q(x')dx'\right]$

Region Ⅱ： $\phi_{\mathrm{II}}(x)=A|Q(x)|^{-1/2}\exp\left[\int_{x_2}^x |Q(x')|dx'\right]+B|Q(x)|^{-1/2}\exp\left[-\int_{x_2}^x |Q(x')|dx'\right]$

Region Ⅲ： $\phi_{\mathrm{III}}(x)=DQ^{-1/2}(x)\exp\left[-i\int_{x_1}^x Q(x')dx'\right]$

$$Q^2=\left(\frac{E}{\hbar c}\right)^2-V\cong\left(\frac{E}{\hbar c}\right)^2-V(x_2)-V'(x_2)(x-x_2)=-V'(x_2)(x-x_2)=\alpha^3(x-x_2)$$

其中$\alpha=[-V'(x_2)]^{1/3}$，令$z=\alpha(x-x_2)$，因此$Q\cong\sqrt{\alpha^3(x-x_2)}=\alpha\sqrt{z}$

$$\int_{x_2}^x Q(x')dx'=\int_{x_2}^x\sqrt{\alpha^3(x'-x_2)}dx'=\sqrt{\alpha^3}\int_{x_2}^x\sqrt{(x'-x_2)}dx'=\frac{2}{3}\sqrt{\alpha^3}(x-x_2)^{3/2}=\frac{2}{3}z^{3/2}$$

$$\phi_{\mathrm{I}}=\frac{1}{\sqrt{\alpha}z^{1/4}}\exp\left[-i\frac{2}{3}z^{3/2}\right]+\frac{C}{\sqrt{\alpha}z^{1/4}}\exp\left[i\frac{2}{3}z^{3/2}\right]$$
$$=\left[\frac{-i\exp(i\pi/4)}{\sqrt{\alpha}z^{1/4}}+iC\frac{\exp(-i\pi/4)}{\sqrt{\alpha}z^{1/4}}\right]\sin\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$
$$+\left[\frac{\exp(i\pi/4)}{\sqrt{\alpha}z^{1/4}}+C\frac{\exp(-i\pi/4)}{\sqrt{\alpha}z^{1/4}}\right]\cos\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$

$Q^2\cong\alpha^3(x-x_2)=\alpha^2 z$ ， $\frac{d^2}{dx^2}=\alpha^2\frac{d^2}{dz^2}$

$\frac{d^2\phi}{dx^2}+Q^2\phi=0 \quad \Rightarrow\frac{d^2\phi_p}{dx^2}+\alpha^2 z\phi_p=0 \quad \Rightarrow\frac{d^2\phi_p}{dz^2}=-z\phi_p$

令 $w=-z$ ， $\frac{d^2}{dz^2}=\frac{d^2}{dw^2}$ ，可得

$$\frac{d^2\phi_p}{dw^2}=w\phi_p \qquad (3.23)$$

(3.23)式為 Airy 方程式，其解為 Airy 函數，當 $w<<0$ 時

$$\phi_p=aA_i(w)+bB_i(w)$$
$$=\frac{a}{\sqrt{\pi}(-w)^{1/4}}\sin\left[\frac{2}{3}(-w)^{3/2}+\frac{\pi}{4}\right]+\frac{b}{\sqrt{\pi}(-w)^{1/4}}\cos\left[\frac{2}{3}(-w)^{3/2}+\frac{\pi}{4}\right]$$
$$=\frac{a}{\sqrt{\pi}z^{1/4}}\sin\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]+\frac{b}{\sqrt{\pi}z^{1/4}}\cos\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$

其中 $A_i(w)$ 與 $B_i(w)$ 為 Airy 函數，$\phi_{\mathrm{I}}=\phi_p$

$$\frac{a}{\sqrt{\pi}z^{1/4}}\sin\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]+\frac{b}{\sqrt{\pi}z^{1/4}}\cos\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$
$$=\left[\frac{-i\exp(i\pi/4)}{\sqrt{\alpha}z^{1/4}}+iC\frac{\exp(-i\pi/4)}{\sqrt{\alpha}z^{1/4}}\right]\sin\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$
$$+\left[\frac{\exp(i\pi/4)}{\sqrt{\alpha}z^{1/4}}+C\frac{\exp(-i\pi/4)}{\sqrt{\alpha}z^{1/4}}\right]\cos\left[\frac{2}{3}z^{3/2}+\frac{\pi}{4}\right]$$

比較可得

$$\begin{cases} a=\sqrt{\frac{\pi}{\alpha}}\left[-i\exp(i\pi/4)+iC\exp(-i\pi/4)\right] \\ b=\sqrt{\frac{\pi}{\alpha}}\left[\exp(i\pi/4)+C\exp(-i\pi/4)\right] \end{cases} \qquad (3.24)$$

$Q\cong\sqrt{\alpha^3(x-x_2)}$ ， $\alpha=\left[-V'(x_2)\right]^{1/3}$ ， $z=\alpha(x-x_2)$ ，$|Q|\cong\sqrt{(-\alpha)^3(x-x_2)}=\alpha\sqrt{-z}$

$$\int_{x_2}^{x}|Q(x')|dx'=\int_{x_2}^{x}\sqrt{(-\alpha)^3(x'-x_2)}dx'=\sqrt{(-\alpha)^3}\int_{x_2}^{x}\sqrt{(x'-x_2)}dx'$$
$$=\frac{2}{3}\sqrt{(-\alpha)^3}(x-x_2)^{3/2}=\frac{2}{3}(-z)^{3/2}$$

$$\phi_{\text{II}}=\frac{A}{\sqrt{\alpha}(-z)^{1/4}}\exp\left[\frac{2}{3}(-z)^{3/2}\right]+\frac{B}{\sqrt{\alpha}(-z)^{1/4}}\exp\left[-\frac{2}{3}(-z)^{3/2}\right]$$

$\frac{d^2\phi_p}{dw^2}=w\phi_p$ ， $w=-z=-\alpha(x-x_2)$ ，當 $w>>0$ 時

$$\begin{aligned}\phi_p&=aA_i(w)+bB_i(w)\\&=\frac{a}{2\sqrt{\pi}w^{1/4}}\exp\left[-\frac{2}{3}w^{3/2}\right]+\frac{b}{\sqrt{\pi}w^{1/4}}\exp\left[\frac{2}{3}w^{3/2}\right]\\&=\frac{a}{2\sqrt{\pi}(-z)^{1/4}}\exp\left[-\frac{2}{3}(-z)^{3/2}\right]+\frac{b}{\sqrt{\pi}(-z)^{1/4}}\exp\left[\frac{2}{3}(-z)^{3/2}\right]\end{aligned}$$

$\phi_{\text{II}}=\phi_p$

$$\begin{aligned}&\frac{a}{2\sqrt{\pi}(-z)^{1/4}}\exp\left[-\frac{2}{3}(-z)^{3/2}\right]+\frac{b}{\sqrt{\pi}(-z)^{1/4}}\exp\left[\frac{2}{3}(-z)^{3/2}\right]\\&=\frac{A}{\sqrt{\alpha}(-z)^{1/4}}\exp\left[\frac{2}{3}(-z)^{3/2}\right]+\frac{B}{\sqrt{\alpha}(-z)^{1/4}}\exp\left[-\frac{2}{3}(-z)^{3/2}\right]\end{aligned}$$

比較可得

$$\begin{cases}a=2\sqrt{\frac{\pi}{\alpha}}B\\ b=\sqrt{\frac{\pi}{\alpha}}A\end{cases}\qquad(3.25)$$

由(3.24)、(3.25)式可得

$$\begin{cases}2B=-i\exp(i\pi/4)+iC\exp(-i\pi/4)\\ A=\exp(i\pi/4)+C\exp(-i\pi/4)\end{cases}\qquad(3.26)$$

$$Q^2=\left(\frac{E}{\hbar c}\right)^2-V\cong\left(\frac{E}{\hbar c}\right)^2-V(x_1)-V'(x_1)(x-x_1)=-V'(x_1)(x-x_1)=-\beta^3(x-x_1)$$

其中 $\beta=[V'(x_1)]^{1/3}$ ，令 $z=\beta(x-x_1)$ ，因此 $Q\cong\sqrt{-\beta^3(x-x_1)}=\beta\sqrt{-z}$

$$\begin{aligned}\int_{x_1}^{x}Q(x')dx'&=\int_{x_1}^{x}\sqrt{-\beta^3(x'-x_1)}dx'=\sqrt{(-\beta)^3}\int_{x_1}^{x}\sqrt{(x'-x_1)}dx'\\&=\frac{2}{3}\sqrt{(-\beta)^3}(x-x_1)^{3/2}=\frac{2}{3}(-z)^{3/2}\end{aligned}$$

$$\begin{aligned}\phi_{\text{III}}&=\frac{D}{\sqrt{\beta}(-z)^{1/4}}\exp\left[-i\frac{2}{3}(-z)^{3/2}\right]\\&=-\frac{iD\exp(i\pi/4)}{\sqrt{\beta}(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]+\frac{D\exp(i\pi/4)}{\sqrt{\beta}(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]\end{aligned}$$

$Q^2\cong-\beta^3(x-x_1)=-\beta^2 z$ ， $\frac{d^2}{dx^2}=\beta^2\frac{d^2}{dz^2}$

$$\frac{d^2\phi}{dx^2}+Q^2\phi=0 \quad \Rightarrow \frac{d^2\phi_p}{dx^2}-\beta^2 z\phi_p=0 \quad \Rightarrow \frac{d^2\phi_p}{dz^2}=z\phi_p$$

當 $z<<0$ 時

$$\begin{aligned}\phi_p &= a'A_i(z)+b'B_i(z)\\ &=\frac{a'}{\sqrt{\pi}(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]+\frac{b'}{\sqrt{\pi}(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]\end{aligned}$$

$\phi_{\text{III}}=\phi_p$

$$\begin{aligned}&\frac{a'}{\sqrt{\pi}(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]+\frac{b'}{\sqrt{\pi}(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]\\ &=-\frac{iD\exp(i\pi/4)}{\sqrt{\beta}(-z)^{1/4}}\sin\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]+\frac{D\exp(i\pi/4)}{\sqrt{\beta}(-z)^{1/4}}\cos\left[\frac{2}{3}(-z)^{3/2}+\frac{\pi}{4}\right]\end{aligned}$$

比較可得

$$\begin{cases} a'=\sqrt{\dfrac{\pi}{\beta}}\left[-iD\exp(i\pi/4)\right] \\ b'=\sqrt{\dfrac{\pi}{\beta}}\left[D\exp(i\pi/4)\right]\end{cases} \qquad (3.27)$$

$$\begin{aligned}\phi_{\text{II}}(x) &= A|Q(x)|^{-1/2}\exp\left[\int_{x_2}^{x}|Q(x')|dx'\right]+B|Q(x)|^{-1/2}\exp\left[-\int_{x_2}^{x}|Q(x')|dx'\right]\\ &=A|Q(x)|^{-1/2}\exp\left[\int_{x_2}^{x_1}|Q(x')|dx'\right]\exp\left[\int_{x_1}^{x}|Q(x')|dx'\right]\\ &\quad +B|Q(x)|^{-1/2}\exp\left[-\int_{x_2}^{x_1}|Q(x')|dx'\right]\exp\left[-\int_{x_1}^{x}|Q(x')|dx'\right]\end{aligned}$$

$Q\cong\sqrt{-\beta^3(x-x_1)}$ ， $\beta=[V'(x_1)]^{1/3}$ ， $z=\beta(x-x_1)$ ， $|Q|\cong\sqrt{\beta^3(x-x_1)}=\beta\sqrt{z}$

$$\begin{aligned}\int_{x_1}^{x}|Q(x')|dx' &= \int_{x_1}^{x}\sqrt{\beta^3(x'-x_1)}dx'=\sqrt{\beta^3}\int_{x_1}^{x}\sqrt{(x'-x_1)}dx'\\ &=\frac{2}{3}\sqrt{\beta^3}(x-x_1)^{3/2}=\frac{2}{3}z^{3/2}\end{aligned}$$

$$\begin{aligned}\phi_{\text{II}} &= \frac{A}{\sqrt{\beta}z^{1/4}}\exp\left[\int_{x_2}^{x_1}|Q(x')|dx'\right]\exp\left[\frac{2}{3}z^{3/2}\right]\\ &\quad+\frac{B}{\sqrt{\beta}z^{1/4}}\exp\left[-\int_{x_2}^{x_1}|Q(x')|dx'\right]\exp\left[-\frac{2}{3}z^{3/2}\right]\end{aligned}$$

$\dfrac{d^2\phi_p}{dz^2}=z\phi_p$ ，當 $z>>0$ 時

$$
\begin{aligned}
\phi_p &= a' A_i(z) + b' B_i(z) \\
&= \frac{a'}{2\sqrt{\pi} z^{1/4}} \exp\left[-\frac{2}{3} z^{3/2}\right] + \frac{b'}{\sqrt{\pi} z^{1/4}} \exp\left[\frac{2}{3} z^{3/2}\right]
\end{aligned}
$$

$\phi_{\text{II}} = \phi_p$

$$
\begin{aligned}
&\frac{a'}{2\sqrt{\pi} z^{1/4}} \exp\left[-\frac{2}{3} z^{3/2}\right] + \frac{b'}{\sqrt{\pi} z^{1/4}} \exp\left[\frac{2}{3} z^{3/2}\right] \\
&= \frac{A}{\sqrt{\beta} z^{1/4}} \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] \exp\left[\frac{2}{3} z^{3/2}\right] \\
&+ \frac{B}{\sqrt{\beta} z^{1/4}} \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] \exp\left[-\frac{2}{3} z^{3/2}\right]
\end{aligned}
$$

比較可得

$$
\begin{cases}
a' = \sqrt{\dfrac{\pi}{\beta}} \left\{ 2B \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] \right\} \\
b' = \sqrt{\dfrac{\pi}{\beta}} \left\{ A \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] \right\}
\end{cases}
\tag{3.28}
$$

由(3.27)式及(3.28)式可得

$$
\begin{cases}
2B \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] = -iD \exp(i\pi/4) \\
A \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] = D \exp(i\pi/4)
\end{cases}
\tag{3.29}
$$

將(3.26)式與(3.29)式聯立

$$
\begin{cases}
2B = -i \exp(i\pi/4) + iC \exp(-i\pi/4) & (3.30) \\
A = \exp(i\pi/4) + C \exp(-i\pi/4) & (3.31) \\
2B \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] = -iD \exp(i\pi/4) & (3.32) \\
A \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] = D \exp(i\pi/4) & (3.33)
\end{cases}
$$

將(3.30)式代入(3.32)式，將(3.31)式代入(3.33)式

$$
\begin{cases}
\left[-i \exp(i\pi/4) + iC \exp(-i\pi/4)\right] \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] = -iD \exp(i\pi/4) \\
\left[\exp(i\pi/4) + C \exp(-i\pi/4)\right] \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] = D \exp(i\pi/4)
\end{cases}
$$

$$
\Rightarrow
\begin{cases}
(1 + iC) \exp\left[-\int_{x_2}^{x_1} |Q(x')| dx'\right] = D \\
(1 - iC) \exp\left[\int_{x_2}^{x_1} |Q(x')| dx'\right] = D
\end{cases}
\tag{3.34}
$$

(3.34)式可再化為一式

$$\left(1+iC\right)\exp\left[-\int_{x_2}^{x_1}\left|Q(x')\right|dx'\right]=\left(1-iC\right)\exp\left[\int_{x_2}^{x_1}\left|Q(x')\right|dx'\right]$$

$$\Rightarrow C=i\left\{\frac{\exp\left[2\int_{x_1}^{x_2}\left|Q(x')\right|dx'\right]-1}{\exp\left[2\int_{x_1}^{x_2}\left|Q(x')\right|dx'\right]+1}\right\} \qquad (3.35)$$

將(3.35)式代入(3.34)式，可得

$$D=\frac{2\exp\left[\int_{x_1}^{x_2}\left|Q(x')\right|dx'\right]}{\exp\left[2\int_{x_1}^{x_2}\left|Q(x')\right|dx'\right]+1} \qquad (3.36)$$

(3.35)式與(3.36)式可合併成一式

$$\left|C\right|^2+\left|D\right|^2=1 \qquad (3.37)$$

(3.37)式即是機率守恆關係式，$\left|C\right|^2$是反射率，相當於第二章的反射率R，$\left|D\right|^2$是透射率，相當於第二章的透射率T。

3.6 反射率與透射率的數值計算與 WKB 近似的結果比較

$R-\zeta$ 及 $T-\zeta$ 關係圖（縱軸座標為機率大小，橫軸座標為ζ）

$m_0=0$，$l=0$

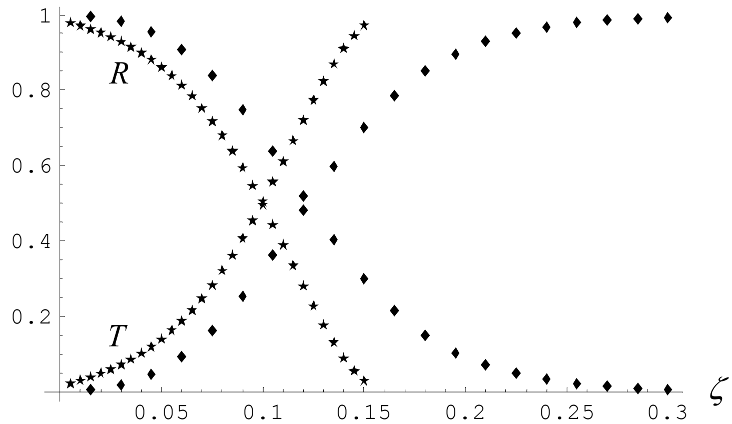


圖 3.3：「◆」符號是利用數值方法求得的$R-\zeta$及$T-\zeta$關係圖，「★」符號是利用 WKB 近似法求得的$R-\zeta$及$T-\zeta$關係圖。

$m_0 = 0$ ， $l = 1$

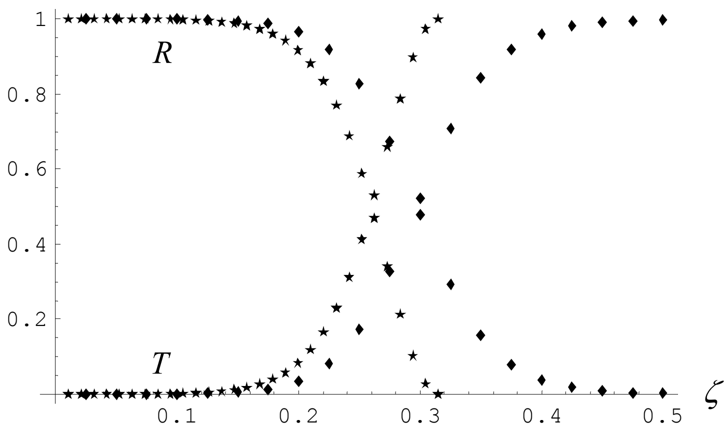


圖 3.4：「◆」符號是利用數值方法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖，「★」符號是利用 WKB 近似法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖。

$m_0 = 0$ ， $l = 2$

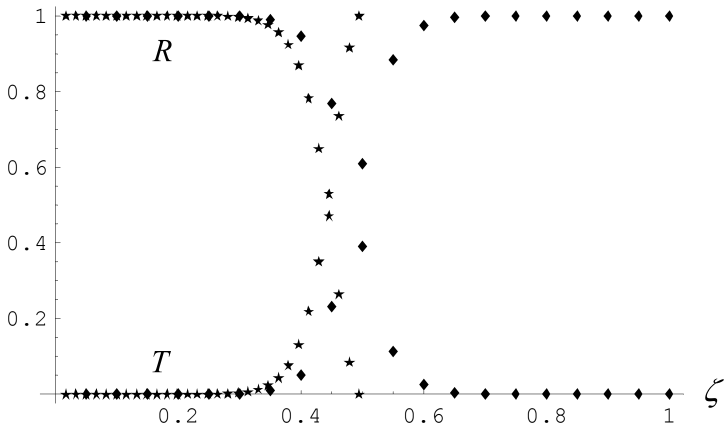


圖 3.5：「◆」符號是利用數值方法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖，「★」符號是利用 WKB 近似法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖。

$$Q=\sqrt{\left(\frac{E}{\hbar c}\right)^2-V}=\sqrt{\left(\frac{E}{\hbar c}\right)^2-\left\{\frac{1}{\left(2GM/c^2\right)^2}\left[\frac{y}{(y+1)^4}+l(l+1)\frac{y}{(y+1)^3}\right]+\left(\frac{m_0c}{\hbar}\right)^2\frac{y}{y+1}\right\}}$$

$$x=\frac{2GM}{c^2}\left[y+1+\ln(y)\right]\text{，}\frac{dx}{dy}=\frac{2GM}{c^2}\left(\frac{y+1}{y}\right)$$

$$\int_{x_1}^{x_2}\left|Q(x')\right|dx'=\int_{y_1}^{y_2}\left|Q(y')\right|\left(\frac{dx'}{dy'}\right)dy'$$

$$=\int_{y_1}^{y_2}\left|\sqrt{\left(\frac{E}{\hbar c}\right)^2-\left\{\frac{1}{\left(2GM/c^2\right)^2}\left[\frac{y'}{(y'+1)^4}+l(l+1)\frac{y'}{(y'+1)^3}\right]+\left(\frac{m_0c}{\hbar}\right)^2\frac{y'}{y'+1}\right\}}\right|\left(\frac{2GM}{c^2}\right)\left(\frac{y'+1}{y'}\right)dy'$$

$$=\int_{y_1}^{y_2}\left|\sqrt{\left(\frac{E}{\hbar c}\right)^2\left(\frac{2GM}{c^2}\right)^2-\left\{\left[\frac{y'}{(y'+1)^4}+l(l+1)\frac{y'}{(y'+1)^3}\right]+\left(\frac{m_0c}{\hbar}\right)^2\left(\frac{2GM}{c^2}\right)^2\frac{y'}{y'+1}\right\}}\right|\left(\frac{y'+1}{y'}\right)dy'$$

$$=\int_{y_1}^{y_2}\left|\sqrt{(2\zeta)^2-\left\{\left[\frac{y'}{(y'+1)^4}+l(l+1)\frac{y'}{(y'+1)^3}\right]+(2\eta)^2\frac{y'}{y'+1}\right\}}\right|\left(\frac{y'+1}{y'}\right)dy'$$

下圖是利用 WKB 近似法所求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖（縱軸座標為機率大小，橫軸座標為 ζ ）

$\eta=1/2$ ，$l=2$

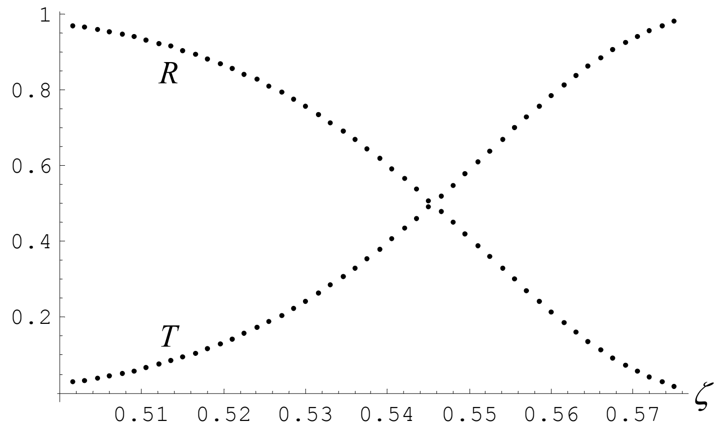


圖 3.6：上圖是利用 WKB 近似法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖。
當 $\zeta\cong 0.55$ 時，$R=T=0.5$ 。

$\eta = 1/2$ ， $l = 3$

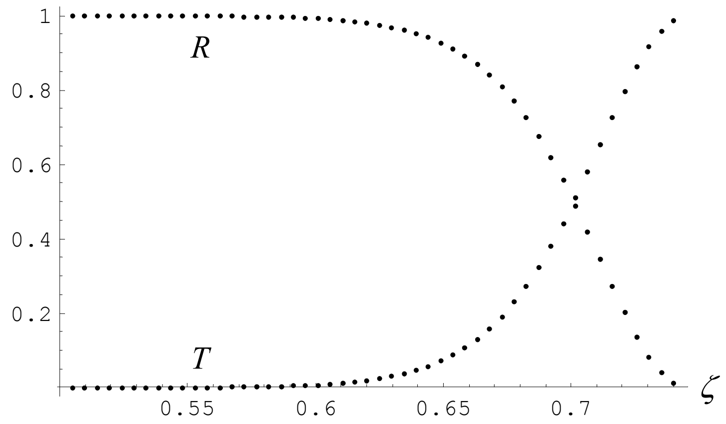


圖 3.7：上圖是利用 WKB 近似法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖。
當 $\zeta \cong 0.7$ 時， $R = T = 0.5$ 。

$\eta = 1/2$ ， $l = 4$

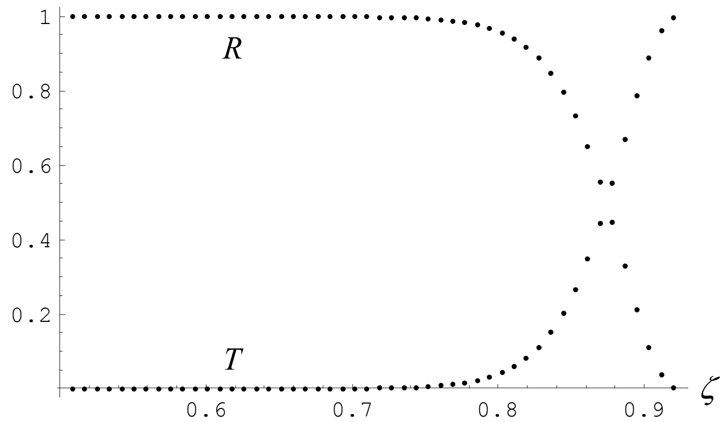


圖 3.8：上圖是利用 WKB 近似法求得的 $R-\zeta$ 及 $T-\zeta$ 關係圖。
當 $\zeta \cong 0.9$ 時， $R = T = 0.5$ 。