2 The case for $k \geq 4$

By using the Method of Infinite Decent, we can prove Theorem 2.1.

Theorem 2.1 *The Diophantine equation* $x^2 + y^2 + z^2 = kxyz$ *has no positive interger solution, where* $k \geq 4$, $k \in \mathbf{N}$.

Proof. First, we claim that the entries of the solution (a, b, c) are all distinct. We suppose that $a = b$, then $2a^2 + c^2 = ka^2c$. Hence $a \mid c$. Let $c = na$, $n \in \mathbf{N}$, we can get $2 + n^2 = nka$. This implies $n \mid 2$, thus $n = 1$ or 2, contradicting to the equation.

Next, let (a, b, c) be a solution. It is easy to see that c and $c' = kab - c$ are actually the roots of the quadratic equation $z^2 - kabz + a^2 + b^2 = 0$ in z. It follows that $cc' = a^2 + b^2$. In particular, $c' > 0$. Thus, $(a, b, kab - c)$ is indeed a solution.

We consider that

$$\begin{aligned}(c-b)(c'-b) &= cc' - (c+c')b + b^2 \\ &= a^2 + b^2 - kab^2 + b^2 \\ &= a^2 + 2b^2 - kab^2 \\ &< 3b^2 - kab^2 \\ &= b^2(3 - ka) < 0\end{aligned}$$

By the Method of Infinite Descent, we complete the proof of Theorem 2.1. ∎