2 Minimax theorems under η-connectedness

Let X be a nonempty set in a topological space, Y a nonempty η-connected set in a vector space for some η, and $f : X \times Y \longrightarrow \mathbb{R}$ be a real-valued function. Then f is said to be *lower semicontinuous on the* η *of* Y if for each $x \in X$ and $y_1, y_2 \in Y$, the function

$$t \longrightarrow f(x, \eta(y_1, y_2, t))$$

is a lower semicontinuous function of t on $[0,\ 1]$. If f is a lower semicontinuous function on Y, then f is clearly lower semicontinuous on the η of Y.

In this section, our main result involves only the η-connectedness instead of convexity. Before proceeding this, we require the following technical lemma of Farka's type.

Lemma 2.1. *Let X be a nonempty compact set of a topological space and let Y be a nonempty η-connected set. Let $f,\ g$ be real-valued functions defined on $X \times Y$ with the following properties:*

(0) *$f(x,y) \le g(x,y)$ for all $(x,y) \in X \times Y$;*
(i) *$f(\cdot,y)$ and $g(\cdot,y)$ are upper semicontinuous on X for each $y \in Y$;*
(ii) *For any $y_0, y_1 \in Y$,*

$$g(x, \eta(y_0, y_1, t)) \le \max\{g(x,y_0), f(x,y_1)\}$$

for all $x \in X$, for all $t \in [0,1]$;
(iii) *f is lower semicontinuous on the η of Y;*
(iv) *For any $y_1, \cdots, y_n \in Y$, any $\lambda \in \mathbb{R}$, the set $\cap_{i=1}^n \{x \in X; g(x,y_i) \ge \lambda\}$ is either connected or empty.*

Then for any $\lambda \in \mathbb{R}$, we have the following alternative:
Either there exists $x_0 \in X$ such that $g(x_0,y) \ge \lambda$ for all $y \in Y$
or there exists $y_0 \in Y$ such that $f(x,y_0) < \lambda$ for all $x \in X$.

Proof. *For each $\lambda \in \mathbb{R}, y \in Y$, let*

$$U_g(y) = \{x \in X; g(x,y) \ge \lambda\},$$

and

$$U_f(y) = \{x \in X; f(x,y) \ge \lambda\}.$$

Fixed $\lambda \in \mathbb{R}$, if for some $y_0 \in Y$, $U_f(y_0) = \emptyset$, then $f(x,y_0) < \lambda$ for all $x \in X$. Thus, we may assume that

$$U_f(y) \neq \emptyset. \tag{1}$$

for all $y \in Y$. We are going to show that

$$\bigcap_{y \in Y} U_g(y) \neq \emptyset.$$

Since X is compact, we need only to show that the family $\{U_g(y)\}_{y \in Y}$ has the finite intersection property, that is,

$$\bigcap_{y \in F} U_g(y) \neq \emptyset \text{ for all finite subsets } F \subset Y.$$

If $|F|$=1. By (1), $U_f(y) \neq \emptyset$ for all $y \in Y$. From condition (0) , $U_g(y) \supset U_f(y)$, so $U_g(y) \neq \emptyset$ for all $y \in Y$.

Let $F = \{y_0, y_1\} \subset Y$, $y_0 \neq y_1$. We want to show that

$$U_g(y_0) \cap U_g(y_1) \neq \emptyset$$

By condition(0) and

$$U_g(y_0) \cap U_f(y_1) = \bigcap_{\epsilon > 0} (U_g(y_0, \lambda - \epsilon) \cap U_f(y_1, \lambda - \epsilon)),$$

it is sufficent to show that for any $\epsilon > 0$,

$$U_g(y_0, \lambda - \epsilon) \cap U_f(y_1, \lambda - \epsilon) \neq \emptyset \tag{2}$$

Fixed $\epsilon > 0$, *for each* $t \in [0,1]$, *consider the following sets*

$$\begin{aligned} S_g(t) &\equiv U_g(\eta(y_0, y_1, t), \lambda - \epsilon) \\ &= \{x \in X; g(x, \eta(y_0, y_1, t)) \geq \lambda - \epsilon\}, \end{aligned}$$

and

$$\begin{aligned} S_f(t) &\equiv U_f(\eta(y_0, y_1, t), \lambda - \epsilon) \\ &= \{x \in X; f(x, \eta(y_0, y_1, t)) \geq \lambda - \epsilon\}. \end{aligned}$$

Notice that, for all $t \in [0,1]$, $S_g(t)$ *is a nonempty compact connected subset of* X *by* (1), *conditions* (i) *and* (iv), *and* $S_f(t)$ *is nonempty closed subset of* X *by* (1) *and condition* (i).

Let

$$A_0 \equiv \{t \in [0,1]; S_f(t) \subset S_g(0)\},$$

and

$$A_1 \equiv \{t \in [0,1]; S_f(t) \subset S_f(1)\},$$

If $S_g(0) \cap S_f(1)$ *were empty set, we claim* (α), (β) *and* (γ) *as follows.*

(α) $A_0 \cap A_1 = \emptyset$.
This is clear by the construction of these two sets.

(β) $A_0 \cup A_1 = [0,1]$.
Indeed, by (ii), *we have, for all* $x \in X$, *for all* $t \in [0,1]$,

$$g(x, \eta(y_0, y_1, t)) \leq \max\{g(x, y_0), f(x, y_1)\}.$$

For each $t \in [0,1]$, *let* $x \in S_f(t)$, *so* $x \in S_g(t)$ *by condition* (0) . *Then* $g(x, \eta(y_0, y_1, t)) \geq \lambda - \epsilon$. *It follows that* $g(x, y_0) \geq \lambda - \epsilon$ *or* $f(x, y_1) \geq \lambda - \epsilon$ *by condition* (ii). *This implies that* $x \in S_g(0)$ *or* $x \in S_f(1)$, *and hence* $S_g(t) \subset S_g(0) \cup S_f(1)$. *Since* $S_g(t)$ *is connected,* $S_g(0)$ *and* $S_f(1)$ *are nonempty closed sets and* $S_g(0) \cap S_f(1) = \emptyset$, *we have either* $S_g(t) \subset S_g(0)$ *or* $S_g(t) \subset S_f(1)$ *and hence* $S_f(t) \subset S_g(0)$ *or* $S_f(t) \subset S_f(1)$. *It follows that* $t \in A_0$ *or* $t \in A_1$. *This shows* (β).

(γ) *Both* A_0 *and* A_1 *are closed subsets of* $[0,1]$.
Let $\{t_n\}$ *be a sequence in* A_0 *with* $t_n \longrightarrow t_0 \in [0,1]$. *Let* $x \in S_f(t_0)$.
Then

$$f(x, \eta(y_0, y_1, t)) \geq \lambda - \epsilon. \tag{3}$$

From (iii), *since* f *is lower semicontinuous on the* η *of* Y, *the function* F *defined by* $F(t) = f(x, \eta(y_0, y_1, t))$ *is lower semicontinuous on* $[0,1]$; *thus* $\liminf_n F(t_n) \geq F(t_0)$. *It follows that there exits* $m \in N$ *such that* $F(t_m) \geq F(t_0)$. *That is,* $f(x, \eta(y_0, y_1, t_m)) \geq f(x, \eta(y_0, y_1, t_0))$. *By* (3), *this implies that* $x \in S_f(t_m)$. *Since* $t_m \in A_0$, *we have* $x \in S_g(0)$. *Therefore,* $S_f(t_0) \subset S_g(0)$. *It follows that* $t_0 \in A_0$, *so* A_0 *is closed. Similarly,* A_1 *is closed.*

From (α), (β), (γ), *we get a contradiction about the connectedness of* $[0,1]$, *since* A_0 *and* A_1 *are nonempty sets* ($0 \in A_0$ *and* $1 \in A_1$). *Therefore,* $S_g(0) \cap S_f(1) \neq \emptyset$. *Hence,* (2) *holds.*

Now, we proceed the proof by induction on $|F|$, *and suppose that* $\cap_{y\in F} U_g(y) \neq \emptyset$ *for all* $F \subset Y$, $2 \leq |F| \leq k$.

Choose $\{y_0, \cdots, y_k\} \subset Y, y_i \neq y_j$ *if* $i \neq j$. *From the above argument, in order to show that* $\cap_{i=0}^{k} U_g(y_i) \neq \emptyset$, *it is sufficent to show, for each* $\epsilon > 0$,*that*

$$U_g(y_0, \lambda - \epsilon) \bigcap U_f(y_1, \lambda - \epsilon) \bigcap (\cap_{i=2}^{k} U_g(y_i, \lambda - \epsilon)) \neq \emptyset. \tag{4}$$

Let $\epsilon > 0$ *and* $H = \cap_{i=2}^{k} U_g(y_i, \lambda - \epsilon)$. *Notice that, by induction hypothesis, since* $\bigcap_{\epsilon>0} [H \cap U_g(y, \lambda - \epsilon)] = (\cap_{i=2}^{k} U_g(y_i)) \bigcap U_g(y) \neq \emptyset$ *for all* $y \in Y$, $H \cap U_g(y, \lambda - \epsilon) \neq \emptyset$ *for all* $y \in Y$. *In particular,*

$$H \cap S_g(t) \neq \emptyset \ \textit{for all} \ t \in [0,1].$$

Also, by (iv), $H \cap S_g(t)$ *is connected for all* $t \in [0,1]$. *For* $t \in [0,1]$, *we let*

$$B_0 \equiv \{t \in [0,1]; H \cap S_f(t) \subset H \cap S_g(0)\},$$

and

$$B_1 \equiv \{t \in [0,1]; H \cap S_f(t) \subset H \cap S_f(1)\}.$$

If $(H \cap S_g(0)) \cap (H \cap S_f(1))$ *were empty set, then by the same argument as above, we would have*

(α') $B_0 \cap B_1 = \emptyset$.

(β') $B_0 \cup B_1 = [0,1]$.

(γ') *Both* B_0 *and* B_1 *are nonempty closed subsets of* $[0,1]$.

This again contradicts the connectedness of $[0,1]$. *Therefore,* (4) *holds and hence* $\cap_{i=0}^{k} U_g(y_i) \neq \emptyset$.

□

Remark 2.2. *The condition (iv) of Lemma 2.1 is a weaker form introducted by Geraghty and Lin [6] than a connected condition used by Terkelsen [7].*

Using the above technique Lemma, we now establish our main result as follow.

Theorem 2.3. *Let* X *be a nonempty compact set of a topological space and let* Y *be a nonempty* η*-connected set. Let* f, g *be real-valued functions defined on* $X \times Y$ *with the following properties:*

(0) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$;

(i) $f(\cdot, y)$ *and* $g(\cdot, y)$ *are upper semicontinuous on* X *for each* $y \in Y$;

(ii) *For any* $y_0, y_1 \in Y$,

$$g(x, \eta(y_0, y_1, t)) \leq \max\{g(x, y_0), f(x, y_1)\}$$

for all $x \in X$, for all $t \in [0,1]$;

(iii) *f is lower semicontinuous on the η of Y;*

(iv) *For any $y_1, \cdots, y_n \in Y$, any $\lambda \in \mathbb{R}$, the set $\cap_{i=1}^{n}\{x \in X; g(x, y_i) \geq \lambda\}$ is either connected or empty.*

Then

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y)$$

Proof. *Let $\lambda \in \mathbb{R}$ with*

$$\sup_{x\in X}\inf_{y\in Y} g(x,y) < \lambda.$$

Thus, there exists $y_0 \in Y$ such that $g(x, y_0) < \lambda$ for all $x \in X$. By Lemma 2.1, there exists $y' \in Y$ such that $f(x, y') < \lambda$ for all $x \in X$. This implies that $\inf_{y\in Y}\sup_{x\in X} f(x,y) < \lambda$. It follows that

$$\inf_{y\in Y}\sup_{x\in X} f(x,y) \leq \sup_{x\in X}\inf_{y\in Y} g(x,y)$$

□

The η-connected property uniformly on Y can be replaced by pointwise η-connectedness defined below. Let Y be a nonempty set. Define

$$G(y_0, y_1) \equiv \left\{ \eta : Y \times Y \times [0,1] \longrightarrow Y; \eta \text{ is continuous and } \begin{array}{l} \eta(y_0, y_1, 0) = y_0 \\ \eta(y_0, y_1, 1) = y_1 \end{array} \right\}$$

We say Y is *pointwise η-connected* if for any $y_0, y_1 \in Y$, there exists η in $G(y_0, y_1)$ such that

$$g(x, \eta(y_0, y_1, t)) \leq \max\{g(x, y_0), f(x, y_1)\}$$

for all $x \in X$, for all $t \in [0,1]$.

By a similar process of proof in Lemma 2.1, we can get the following theorem immediately.

Theorem 2.4. *Let X be a nonempty compact set of a topological space, let Y be a nonempty pointwise η-connected set. Let f, g be real-valued functions defined on $X \times Y$ with the following properties:*

(0) *$f(x,y) \leq g(x,y)$ for all $(x,y) \in X \times Y$;*

(i) *$f(\cdot, y)$ and $g(\cdot, y)$ are upper semicontinuous on X for each $y \in Y$;*

(ii) *For any $y_0, y_1 \in Y$, there exists η in $G(y_0, y_1)$ such that*

$$g(x, \eta(y_0, y_1, t)) \leq \max\{g(x, y_0), f(x, y_1)\}$$

for all $x \in X$, for all $t \in [0,1]$, and

f is lower semicontinuous on the η of Y

(iii) *For any $y_1, \cdots, y_n \in Y$, any $\lambda \in \mathbb{R}$, the set $\cap_{i=1}^{n}\{x \in X; g(x, y_i) \geq \lambda\}$ is either connected or empty. Then for any $\lambda \in \mathbb{R}$, we have the following alternative:*

Either there exists $x_0 \in X$ such that $g(x_0, y) \geq \lambda$ for all $y \in Y$
or there exists $y_0 \in Y$ such that $f(x, y_0) < \lambda$ for all $x \in X$.

Theorem 2.5. *Let X be a nonempty compact set of a topological space, let Y be a nonempty pointwise η-connected set. Let $f,\ g$ be real-valued functions defined on $X \times Y$ with the following properties:*

(0) $f(x,y) \leq g(x,y)$ *for all* $(x,y) \in X \times Y$*;*

(i) $f(\cdot,y)$ *and* $g(\cdot,y)$ *are upper semicontinuous on* X *for each* $y \in Y$*;*

(ii) *For any* $y_0, y_1 \in Y$*, there exists* η *in* $G(y_0,y_1)$ *such that*

$$g(x,\eta(y_0,y_1,t)) \leq \max\{g(x,y_0), f(x,y_1)\}$$

for all $x \in X$*, for all* $t \in [0,1]$*, and*

$$f \text{ is lower semicontinuous on the } \eta \text{ of } Y$$

(iii) *For any* $y_1, \cdots, y_n \in Y$*, any* $\lambda \in \mathbb{R}$*, the set* $\cap_{i=1}^{n}\{x \in X; g(x,y_i) \geq \lambda\}$ *is either connected or empty.*

Then

$$\inf_{y \in Y} \sup_{x \in X} f(x,y) \leq \sup_{x \in X} \inf_{y \in Y} g(x,y)$$