Learning Goals and Effective Mathematics Teaching: What Can We Learn From Research?

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Teaching is a complex endeavor and individual teaching behaviors that promote student learning are hard to identify. In this article, we discuss two methods of gaining advice to improve mathematics teaching and explore in some detail the advice that can be acquired from examining research studies that focus on teaching.

An important truth about the effectiveness of instructional methods is that particular methods are not, in general, effective or ineffective. Particular methods are more or less effective for helping students achieve particular learning goals. Even when the learning goals are specified, the complex dynamics at play in a classroom make it difficult to document the best instructional methods for helping students achieve them. Despite this reality, some teachers are particularly effective in promoting student learning year after year. What is it that these teachers do to be effective and to remain effective at promoting high levels of student learning?

The ways of determining what advice to give to teachers when helping them improve their mathematics teaching can be classified into two general approaches: (1) wisdom of practice and (2) insights from research. Wisdom of practice includes gathering advice from what practicing teachers learn about "what works for them" from their years of teaching experience in classroom settings. This wisdom is often passed on to other teachers formally and informally. For example, teaching experience is often helpful in choosing appropriate tasks, conveying tasks to students in

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a way that stimulates interest, and maintaining students' engagement in tasks. These are all activities that teachers experiment with until they find those that work best for their students and their situations. A major problem with sharing such wisdom of practice, however, is that what works for one teacher may not work for another teacher because the discoveries were associated with a particular group of students and in a specific learning context. In other words, the advice may work only with a specific teaching method with a particular type of learner in a school with a unique set of curriculum materials and a special set of learning objectives.

Other sources of wisdom of practice may come from mathematicians known for their expertise in mathematics who share their intuitive observations of teaching well. One case in point is George Polya's contribution to the process of problem solving as set forth in his book, *How To Solve It*. In this book, Polya characterizes in a useful way the methods people use to solve problems. His work then goes on to include advice on how to teach problem solving. Unfortunately the quality of the advice given in this manner seldom reaches the quality standard we find in Polya's work.

In contrast to advice from the wisdom of practice, insights gained from research on mathematics teaching are the product of empirical studies conducted in a wide variety of classrooms with many teachers and the conclusions drawn are supported by the data collected. Although it is obvious that many interactions are involved in the teaching process and no two classrooms operate identically, it is the case that patterns of teaching effectiveness can be detected across research studies that used different research designs and procedures. These patterns provide insight into features of teaching that regularly produce positive effects on student learning when examined within teaching systems that have a particular set of learning goals. There may be value to both approaches of gathering information for giving advice to teachers, but the focus of this article is on the insights provided by research.

The process we used to select the useful features of teaching to report in this article was based on identifying patterns of positive results within studies associated with two particular learning goals: skill efficiency or conceptual understanding. We choose these two broad learning goals because they are at a level general enough to find many relevant studies of classroom learning and teaching and because they are of current interest across curricula and cultures. Furthermore, these learning goals have a substantial amount of research data that point to effective features of mathematics instruction in each area. In the following, we draw heavily from the work of Hiebert and Grouws (2007) with the realization that the process of identifying studies to examine was limited to considering only those research studies published in journals printed in English.

Skill Efficiency

Skill efficiency refers to the rapid, smooth, and accurate execution of mathematical procedures (Gagne, 1985). Findings from a number of studies reinforce the following claim: mathematics teaching that facilitates skill efficiency is rapidly paced and includes frequent teacher modeling of procedures, the use of many teacher directed product-type questions, and the making of smooth transitions from demonstration of procedures to substantial amounts of error free practice. The teacher plays a central role in organizing, pacing, and presenting information in this type of teaching in order to meet a well-defined learning goal.

Conceptual Understanding

Conceptual understanding refers to the construction of relationships among mathematical facts, procedures, and ideas (Brownell, 1935; Davis, 1984; Hiebert and Carpenter, 1992). From the examination of the research literature, we found two features of instruction that are likely to help students develop conceptual understanding: (a) explicitly attending to connections among facts, procedures, and ideas; and (b) encouraging students to wrestle with important mathematical ideas in an intentional and conscious way. The first feature might sound obvious because it almost is a restatement of the earlier definition of conceptual understanding. In essence, the claim says that if instruction aims to help students develop conceptual understanding, then it must make explicit, at some point and in some way, the critical relationships that lie at the heart of such understanding. This might include discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas.

The second feature associated with increasing students' conceptual understanding is allowing students to struggle with important mathematical ideas. In this context, we do not use the word struggle to mean needless frustration from working on items that present extreme challenge beyond the students' reach. Rather, the struggle we refer to implies that students expend the effort necessary to make sense of the mathematics utilizing tasks or problems chosen to be within the reach of their ability. This is the opposite of simply presenting information to be memorized or asking students to practice only what has been demonstrated. Allowing students to struggle means that the teacher does not step in and do the mathematical work for them, but rather allows the students to grapple with the problems before discussion of possible solution methods and solutions.

Not a Simple Dichotomy

The features described here that promote skill efficiency are quite different from those that promote conceptual understanding. However, the research literature indicates that the development of skills and conceptual understanding is not a simple dichotomy. In fact, results of studies focused on developing conceptual understanding also reported improvement in skill development. On the other hand, what effects the effective features of teaching associated with developing students' skill efficiency have on conceptual understanding is unclear at this point in time. In the next section, we discuss each feature of promoting conceptual understanding in more detail within the context of an example.

Elementary School Example

In the Sieve of Eratosthenes activity (see Appendix A), students are often led through the activity step by step and at the end the teacher proclaims that the numbers not crossed out are special and are called prime numbers. This is an activity that if used in this way does <u>not</u> promote conceptual understanding. In contrast, consider the Factor Game (see Appendix B). The students playing this game very quickly learn on their own that some numbers are special (the prime numbers). Teachers give the label to students <u>after</u> they have discovered the concept in a meaningful way. This is an illustration of teaching where students are provided an opportunity to make sense of the mathematics where the instructional goal is conceptual development.

Secondary School Example

Considering the following problem:

> Find two numbers such that the positive proper divisors sum to be the number itself.
>
> Solution:
>
> $6 = 1 + 2 + 3$

$28 = 1 + 2 + 4 + 7 + 14$

When solving this problem, student solutions can be labeled as “perfect numbers.” Beyond practicing with basic division skills, the problem does little to develop students’ conceptual understanding.

In contrast, consider the following example.

> Given any nine integers, show that you can choose two of them whose difference is a multiple of 8.
>
> One possible solution: Notice that any number divided by 8 has one of eight remainders, 0 through 7. Since there are 9 integers but only 8 possible remainders, two of the numbers when divided by 8 have the same remainder, say, r. Thus, through some symbolic representation and algebraic manipulation, we can show that the two numbers can be expressed as $8m + r$ and $8n + r$, with m and n being integers. Their difference is $8(m – n)$, which is divisible by 8.

Clearly this task provides more of an intellectual challenge than does the previous example of finding perfect numbers. If the task proves to be too challenging, then the teacher can be prepared to provide a student with a helpful hint, but not step-by-step solution help. Notice that within both grade level examples, it is important to allow students to try the problems for an ample amount of time before providing any clues or hints to possible solution methods. In order to continue fostering conceptual development, the teacher can facilitate student discussion of the task, its various solution paths, and valid solutions to ensure that the mathematics embedded within the task is brought to the surface and that every student develops an understanding of it. This is an illustration of providing students an opportunity to struggle with challenging mathematical tasks where the instructional goal is conceptual development.

Summary

The teacher’s role in directing the development of students’ mathematical knowledge is indeed complex. The responsibility is difficult to fulfill because it involves simultaneously fostering students’ knowledge accumulation and developing their mathematical abilities. A teacher must bring these together while considering students’ abilities and backgrounds. However, our analysis of the research literature indicates that the instructional features we have identified for helping students acquire skill efficiency and conceptual understanding are sufficiently documented that they warrant action. Studies have shown that rapidly paced instruction with frequent teacher modeling of procedures and many teacher directed product-type questions is effective if the learning goal is skill efficiency. On the other hand, if the learning goal is conceptual understanding, the research evidence points to instruction that explicitly draws attention to connections among facts, procedures, and ideas and encourages students to wrestle with the important mathematical ideas in an intentional way. We believe these features can be accepted with sufficiently high levels of confidence that they can be used to inform policy and influence classroom practice. Realizing that each class is different with regard to socioeconomic settings and cultural context, we encourage teachers to try these features in their classrooms with the possibility of conducting action research at the same time. Not only can teachers improve their own practice in this way, but they can also share with colleagues more broadly by later reporting their experience with the features.

References

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Appendix A

Sieve of Eratosthenes

	2	3	~~4~~	5
~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15
~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25
~~26~~	27	~~28~~	29	~~30~~

1. Make a chart from including numbers from 2 to a specified integer *n*. (n = 30 in this example)
2. Cross out all multiples of 2 not including 2.
3. Cross out all multiples of 3 not including 3.
4. Cross out all multiples of 5 not including 5.
5. The next number left is 7. Since 7 is larger than $\sqrt{n}$ or, in this case, $\sqrt{30}$, this means all multiples of 7 have been crossed off.
6. The remaining numbers are the prime numbers less than *n*.

Appendix B

The Factor Game Board

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25
26	27	28	29	30

1. Player A chooses a number on the game board by coloring it.
2. Using a different color, Player B colors all the proper factors of Player A's number. The proper factors of a number are all the factors of that number, except the number itself. For example, the proper factors of 12 are 1, 2, 3, 4, and 6. Although 12 is a factor of itself, it is not a proper factor.
3. Player B colors a new number, and Player A colors all the factors of the number that are not already colored.
4. The players take turns choosing numbers and coloring factors. If a player chooses a number that has no factors left that have not been colored, that player loses a turn and does not get the points for the number colored.
5. The game ends when there are no numbers remaining with uncolored factors.
6. Each player adds the numbers that are colored with his or her color. The player with the greater total is the winner.

Retrieved from http://connectedmath.msu.edu/CD/Grade6/FactorGame/