A Random Fixed Point Theorem for Caristi Type Mappings on Metric Spaces

by

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1. Introduction

J. Caristi [3] considers the existence of fixed point for some selfmapping \( f \) on a complete metric space \((X,d)\) satisfying that

\[
d(x, f(x)) \leq \phi(x)-\phi(f(x))
\]

for all \( x \in X \), where \( \phi \) is a nonnegative valued function on \( X \). After [4] having appeared, many papers on fixed point theory are related to mapping with property (1) such as [2], [5], [6], etc..

In this paper we employ our technique in [7] to get existence theorems for a commutative family \( \mathcal{H} \) of selfmappings on a complete metric space \( X \) if \( f \) satisfies (1) for all \( f \in \mathcal{H} \). As an application of our result we show a random fixed point theorem which generalizes a result of Bharucha-Reid [1].

2. Main Result

Throughout this paper we let \((X,d)\) denote a complete metric space. We shall define some notions to be used in the sequel.

Definition 2.1. Let \( f \) be a selfmapping on \( X \). If there is a nonnegative valued function on \( X \) satisfying (1) we shall call \( f \) a selfmapping of Caristi type on \( X \).

Analogous to the above definition we may define the following:

Definition 2.2 Let \( \mathcal{H} \) be a family of selfmappings on \( X \) if there is a nonnegative valued function \( \phi \) on \( X \) satisfying (1) for all \( f \in \mathcal{H} \). We call \( \mathcal{H} \) a family of selfmappings of Caristi type with respect to \( \phi \) on \( X \).

If \( \mathcal{H} \) is a (commutative) family of selfmappings of Caristi type with respect to \( \phi \) and
$S = f_n f_{n-1} \ldots f_1 \in N, f_k \in \mathcal{H}$, for all $k = 1, 2, \ldots, n$.

For any $x \in X$, $f_1, f_2, \ldots, f_n \in \mathcal{H}$, we define

$x_0 = x$ and $x_k = f_k x_{k-1}$ for all $k = 1, 2, \ldots, n$

then

$$d(x_0, x_n) \leq \sum_{k=1}^{n} d(x_{k-1}, x_k)$$

$$\leq \sum_{k=1}^{n} (\phi(x_{k-1}) - \phi(x_k))$$

$$\leq \phi(x_0) - \phi(x_n).$$

Hence, $S$ is a (commutative) semigroup of selfmappings of Caristi type with respect to $\phi$ on $X$.

In particular, if $f$ is of Caristi type with respect to $\phi$, then $\{ f^n, n \in N \}$ is a semigroup of mappings of Caristi type with respect to $\phi$.

It follows from the above remark that we may assume that every family of mappings of Caristi type is a semigroup. If $S$ is a commutative semigroup of selfmappings of Caristi type on $X$, we may define a partial order $\preceq$ on $S$ as follows:

(2) $f \preceq g$ if and only if there is an $h \in S$ with $g = hf$.

If is clear that $(S, \preceq)$ forms a directed set and for each $x \in X \{ fx; f \in S \}$ forms a net, in fact, we have the following

Lemma 2.3. Let $S$ be a commutative semigroup of selfmappings of Caristi type with respect to $\phi$ on $X$. Then for each $x \in X$ we have that $\{ fx; f \in S \}$ converges.

Proof. For $x \in X$ let $r = \inf \{ \phi(fx); f \in S \}$. Then for any $c > 0$, there is an $f_0 \in S$ with $\phi(f_0 x) < r + c$.

By our definition that for $g \in S$ and $f_0 \preceq g$ we have

$$d(f_0 x, gx) \leq \phi(f_0 x) - \phi(g x) < c$$

Hence $\{ fx; f \in S \}$ forms a Cauchy net. By the completeness of $X$ that $\{ fx; f \in S \}$ converges.

With continuity of $f$, we have the following

Theorem 2.4. Let $S$ be a commutative semigroup of selfmappings of Caristi type on $X$. If for each $f \in S$ the mapping $f$ is continuous on orbit $O_x = \{ fx; f \in S \}$ for some $x \in X$ then $y$ is a common fixed point for $S$, where $y$ is the limit of net
Proof. Let \( h \in S \). By the continuity of \( h \) on \( O_x \), we have that for any \( c > 0 \) there is a positive number \( e < c \) such that \( z \in O_x \) and \( d(y, z) < e \) implies \( d(hy, hz) < c \).

Since \( y \) is the limit of \( \{ fx; f \in S \} \) we can find \( f_0 \in S \) with \( g \geq f_0 \) implies that \( d(y, gx) < e \) and then
\[
    d(hy, y) \leq d(hy, hgx) + d(hgx, y) < 2c.
\]
Hence \( hy = y \) for all \( h \in S \).

As a consequence of the above Theorem and Remark, we have the following Corollary 2.5. If \( f \) is a continuous selfmapping of Caristi type on \( X \) then \( f \) have a fixed point. Moreover, for such \( x \in X \) the sequence \( \{ f^n x; n \in \mathbb{N} \} \) converges to a fixed point of \( f \).

Now if \( (M, m, \mu) \) is a measure space we call a measurable function of \( M \) into \( X \) as a random variable on \( X \). If an operator
\[
    T: M \times X \to X
\]
satisfies that
\[
    (i) \ T(\cdot, x) \text{ is a random variable on } X \text{ for all } x \in X, \text{ and}
    \]
\[
    (ii) \ T(\omega, \cdot) \text{ is a selfmapping of Caristi type on } X \text{ for almost all } \omega \in M.
\]
We call \( T \) a random operator of Caristi type on \( X \).

2.6. Theorem. Let \( T: M \times X \to X \) be a random operator of Caristi type. Then there exists a random variable \( y \) on \( X \) such that
\[
    T(\omega, y(\omega)) = y(\omega)
\]
for almost all \( \omega \in M \).

Proof. Let \( \omega \) be an element in \( M \) such that \( T(\omega, \cdot) \) is of Caristi type on \( X \). Define
\[
    f_\omega(x) = T(\omega, x) \text{ for all } x \in X,
\]
then by the previous Corollary that the sequence \( \{ f^n_\omega(x) \} \) converges to some \( y(\omega) \) in \( X \) with \( f_\omega(y(\omega)) = y(\omega) \). If
\[
    x_n(\omega) = f^n_\omega(x), \ n \in \mathbb{N}
\]
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then by the fact that $x_n$ is sequence of random variables on $X$. Then $y(\omega) = \lim x_n(\omega)$, $y$ is also a random variable on $X$ and $f(\omega, y(\omega)) = y(\omega)$ for almost all $\omega \in M$. 


REFERENCES


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〔中文摘要〕

本文考慮一隨機Caristi型函數族之共隨機定點問題，特別，本文推廣了A. T. Bharucha-Reid在隨機積分方程中所提出之相關定理。