On a Generalized Minimax Inequality

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In this paper let $X$ be a convex set of a $(T_2)$ topological vector space and $Y$ an arbitrary $(T_2)$ topological space. We denote $K(X, Y)$ the family of all closed mappings of $X$ into $Y$ with a relatively compact image. A mapping $h$ of $Y$ into $R \cup \{+\infty\}$ is said to be lower semicontinuous if for each $t \in R$ the set \( \{y \in Y; h(y) \leq t\} \) is closed and a mapping $k$ of $X$ into $R \cup \{+\infty\}$ is said to be quasiconcave if for each $t \in R$ the set \( \{x \in X; k(x) > t\} \) is convex.

The main result of this study is as follows:

Theorem. Let $X$ be a convex subset of a topological vector space, $Y$ a topological space. If $f$, $g$ are two mappings of $X \times Y$ into $\bar{R} = R \cup \{+\infty\}$ satisfying the following three conditions:

1. $g(x, \cdot)$ is lower semicontinuous for all $x \in X$,
2. $f(\cdot, y)$ is quasiconcave for all $y \in Y$, and
3. $g(x, y) \leq f(x, y)$ for all $x \in X$ and all $y \in Y$, then for any $s \in K(X, Y)$,

\[ \inf_{y \in Y} \sup_{x \in X} g(x, y) \leq \sup_{x \in X} f(x, s(x)) \]

Special case of the above theorem were proved by Fan [3], Yen [6] and Lassonde [5]. Here we also established a geometric formulation of the above theorem.

Proof. Let $s \in K(X, Y)$. Without loss of the generality we may assume that $\sup f(x, s(x)) = \beta < +\infty$. Define for each $x \in X$ that the sets
\[ F(x) = \{ x' \in X; f(x, s(x')) \leq \beta \}, \]
\[ G(x) = \{ x' \in X; g(x, s(x')) \leq \beta \}. \]

Then by (3) and the definition of \( \beta \) that
\[ G(x) \supset F(x) \neq \emptyset \text{ for all } x \in X. \]

It due to (2) that for any \( x_1, x_2, \ldots, x_k \in X \) we have
\[ \bigcup_{i=1}^{k} G(x_i) \supset \bigcup_{i=1}^{k} F(x_i) \supset \text{conv } \{ x_1, x_2, \ldots, x_k \}. \]

It follows from (1) and the closedness of \( s \) that
\[ G(x) \text{ is closed for all } x \in X. \]

Hence by the K.-K.-M. Theorem [4] that for any \( x_1, x_2, \ldots, x_n \in X, \)
\[ \bigcap_{i=1}^{n} C(x_i) \neq \emptyset. \]

Let \( Y_0 = s(X) \). Define for each \( x \in X \), the set
\[ G_1(x) = \{ y \in Y_0; g(x, y) \leq \beta \} = \{ y \in Y; g(x, y) \leq \beta \} \cap Y_0. \]

Then by (1) that \( G_1(x) \) is compact. We assert that \( \bigcap_{x \in X} G_1(x) \neq \emptyset \). For if not, there are \( x_1, x_2, \ldots, x_n \in X \) with \( \bigcap_{i=1}^{n} G_1(x_i) = \emptyset \), therefore
\[ \emptyset = \bigcap_{i=1}^{n} \{ y \in Y_0; g(x_i, y) \leq \beta \} \]
\[ \supset \bigcap_{i=1}^{n} \{ y \in s(X); g(x_i, y) \leq \beta \} \]
\[ \supset s \left( \bigcap_{i=1}^{n} G(x_i) \right). \]
This is a contradiction to \( \bigcap_{i=1}^{n} G(x_i) \neq \phi \). Hence \( \bigcap_{x \in X} G_1(x) \neq \phi \), that is, there is an \( y_0 \in Y \) with \( \sup_{x \in X} g(x,y_0) \leq \beta \).

Remarks: (i) For the case that \( X = Y \) and \( s(x) \equiv x \) the above minimax inequality was proved by Fan [3] and Yen [6].

(ii) For the case that \( f = g \), and the mapping \( s: X \to Y \) is continuous with a relatively compact image was proved by Lassonde [5].

For the geometric formulation of the above theorem is the following Proposition. If \( A \subset X \times Y \) satisfies that

(4) \[ [x \in X; (x,y) \in A] \text{ is convex for all } y \in Y \]

and there is a set \( B \subset A \) with

(5) \[ [x \in X; (x,y) \in B] \neq \phi \text{ for all } y \in Y \text{ and } \]

(6) \[ [y \in Y; (x,y) \in B] \text{ is open in } X, \]

then for each \( s \in K(X,Y) \) there is an \( x_0 \in X \) with

\[ (x_0,s(x_0)) \in A. \]

Proof. For \( C \subset X \times Y \), define

\[ I_C(z) = \begin{cases} 1 & \text{if } z \in C, \\ 0 & \text{if } z \notin C. \end{cases} \]

Then the conditions (4), (6) and \( B \subset A \) imply that the mappings \( f = I_a \) and \( g = I_B \) satisfy the conditions (1), (2) and (3) of our theorem, hence by (5) we have

\[ 1 = \inf \sup_{y \in Y} I_B(x,y) \leq \sup_{x \in X} I_A(x,s(x)), \]

and then there is an \( x_0 \in A \) with \( (x_0, s(x_0)) \in A. \)

Remark. The above result is closely related to a lemma of Fan [2], and for the case that \( B = A \), the above result is proved by Deguire and Granas [1].
REFERENCES


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〔中文摘要〕

在本文中設 X 爲拓撲問量空間上凸集，Y 爲拓撲空間而 K ( X , Y ) 表由 X 映至 Y 且有相對緊緻值域之閉函數族，而證得了下列定理：

定理：若函數 f , g : X × Y → R ∪ { +∞} 滿足下列三條件：
(1) 對每 x ∈ X , g ( x , ﹒ ) 為下半連続函數，
(2) 對每 y ∈ X , f ( ﹒ , y ) 為擬凹函數，且
(3) 對每 x ∈ X , 每 y ∈ X , g ( x , y ) ≤ f ( x , y )，

則對 s ∈ K ( X , Y ) 恒有

\[ \inf_{y \in Y} \sup_{x \in X} g( x , y ) \leq \sup_{x \in X} f( x , s(x) ) \]

特別情形導致 Fan 及 Yen 大中取小不等式，同時亦導得 Lassonde 之結果。