Forward and backward self-similar solutions for a nonlinear parabolic equation

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1 Introduction

We study the following nonlinear parabolic equation:

\[ u_t = u^\sigma(\Delta u + u^p), \quad x \in \mathbb{R}^n, \quad t > 0, \]  

(1.1)

where \( \sigma \in \mathbb{R}, \ p > 1, \) and \( n \geq 1. \) The equation (1.1) has been extensively studied for the past years, see, for example, [1], [3], [8] for \( \sigma = 0; \) [6] for \( \sigma < 0; \) [2] for \( \sigma \in (0, 1); \) [5], [4] for \( \sigma = 1. \) In all of these works, much attentions have been paid to the blow-up behaviours of solutions in order to understand the mechanism of thermal runway in combustion problem.

We are interested in the global and non-global existence of positive solutions of (1.1) for \( \sigma > 1. \) In particular, we study the existence of forward and backward self-similar positive solutions of (1.1) in the forms

\[ U(x, t) = (t + 1)^{-\alpha}\varphi(\frac{|x|}{(t + 1)^{\beta}}), \]

(1.2)

\[ V(x, t) = (T - t)^{-\alpha}\varphi(\frac{|x|}{(T - t)^{\beta}}), \]

(1.3)

where \( T > 0 \) is given and the similarity exponents are necessarily given by

\[ \alpha = \frac{1}{p + \sigma - 1}, \quad \beta = \frac{p - 1}{2}\alpha. \]

We set \( \xi = |x|/(t + 1)^{\beta}. \) It follows that \( U \) satisfies (1.1) if and only if \( \varphi \) satisfies the following equation

\[ \varphi'' + \frac{n - 1}{\xi}\varphi' + \varphi^p + \alpha\varphi^{1-\sigma} + \beta\xi\varphi^{-\sigma}\varphi' = 0, \quad \xi > 0, \]

(1.4)
and \( \varphi'(0) = 0 \). Similarly, set \( \xi = |x|/(T-t)^\beta \). It follows that \( V \) satisfies (1.1) if and only if \( \varphi \) satisfies the equation
\[
\varphi'' + \frac{n-1}{\xi} \varphi' + \varphi - \alpha \varphi^{1-\sigma} - \beta \xi \varphi^{-\sigma} \varphi' = 0, \quad \xi > 0,
\]
and \( \varphi'(0) = 0 \).

In Section 2, following [5], we shall prove that the solution of (1.4) with initial condition
\[
\varphi'(0) = 0, \varphi(0) = \eta,
\]
exists globally for any \( \eta > 0 \). Also, we prove that \( \varphi(\xi) \) and \( \varphi'(\xi) \) tend to 0 as \( \xi \to \infty \).

Next, we study the equation (1.5) with initial condition (1.6) in Sections 3 and 4 for \( \sigma \in (1,2) \). In Section 3, we study the one-dimensional case. The multiple-dimensional case is treated in Section 4. We show that, for \( n = 1 \), (1.5)-(1.6) has at least \( N-1 \) distinct positive global solutions such that \( \varphi(\xi) \to 0 \) as \( \xi \to \infty \), where \( -N \) is the largest integer which is less or equal to \(- (p + \sigma - 1)/(p-1) = -1/(2\beta) \). Set \( p_c(n) = (n + 2)/(n - 2) \) for \( n \geq 3 \) and \( p_c(2) = \infty \). In parallel to Section 3 of [4], we can also derive that, for \( n \geq 2 \) and \( 1 < p < p_c(n) \), (1.5)-(1.6) has at least \( N_1 \) distinct positive global solutions such that \( \varphi(0) > \kappa \) and \( \varphi(\xi) \to 0 \) as \( \xi \to \infty \), where \( N_1 \) is the largest integral which is less than or equal to \( N/2 \) and \( \kappa = \alpha^\alpha \).

In Section 5, we study the asymptotic behaviors of bounded global solutions of (1.4) and (1.5) as \( \xi \to \infty \). We shall show that for any bounded global solution \( \varphi(\xi) \) of (1.4) or (1.5) the limit
\[
\lim_{\xi \to \infty} \{ \xi^{\alpha/\beta} \varphi(\xi) \} = A
\]
exists and \( A > 0 \). It follows that
\[
\lim_{t \to T^-} V(x,t) = A|x|^{-\alpha/\beta}, \quad x \neq 0
\]
for any nonconstant bounded global solution of (1.5).

From these results there always exists a symmetric positive monotone self-similar solution \( V \) of (1.1) in the form (1.3) such that \( V \) blows up only at single point \( x = 0 \) at \( T \) for a given finite time \( T \). Note that \( N = 2 \) for \( p \geq \sigma + 1 \) and \( N \geq 3 \) for \( p \in (1, \sigma + 1) \). Also, there are some other self-similar single-point blow-up patterns with different oscillations, if \( p \in (1, \sigma + 1) \).

It is also interesting to remark that for any solution \( \varphi \) of (1.4), (1.6) the corresponding forward self-similar solution \( U \) defined by (1.2) exists globally. Note that the initial data \( u_0(x) := U(x,0) = \varphi(|x|) \) satisfies
\[
\lim_{|x| \to \infty} |x|^{\alpha/\beta} u_0(x) = A.
\]