(A note on gradient estimate for the equation associated to the

\( p \)-Laplace operator)

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A note on gradient estimate for the equation associated to the $p$-Laplace operator

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Abstract

In this paper, we study $p$-Laplace operators on complete noncompact manifolds. According to Kotschwar-Ni gradient estimat for positive $p$-harmonic functions, we extend their result to more general equation associated to the $p$-Laplace operator whenever the sectional curvature of $M$ has lower bound.

Key words

$p$-Laplace operator \ gradient estimate
1 Introduction

A real-valued $C^3$ function on a Riemannian $m$-manifold $M$ with a Riemannian metric $\langle \ , \ \rangle$ is said to be strongly $p$-harmonic if $u$ is a (strong) solution of the $p$-Laplace equation (1), $p > 1$,

(1) \[ \Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right) = 0. \]

where $\nabla u$ is the gradient vector field of $u$ on $M$, and $|\nabla u| = \langle \nabla u, \nabla u \rangle^{\frac{1}{2}}$.

A function $u \in W^{1,p}_{\text{Loc}}(M)$ is said to be weakly $p$-harmonic if $u$ is a (Sobolev) weak solution of the $p$-Laplace equation (1), i.e.

\[ \int_M |\nabla u|^{p-2} \langle \nabla u, \nabla \phi \rangle \, dv = 0 \]

holds for every $\phi \in C_0^\infty(M)$, where $dv$ is the volume element of $M$.

The $p$-Laplace equation (1) arises as the Euler-Lagrange equation of the $p$-energy $E_p$ functional given by $E_p(u) = \int_M |\nabla u|^p \, dv$. Uraltseva [20], Evans [5] and Uhlenbeck [21] proved that weak solutions of the equation (1) have Hölder continuous derivatives for $p \geq 2$. Tolksdorff [19], Lewis [12] and DiBenedetto [4] extended the result to $p > 1$. In fact, weak solutions of (1), in general do not have any regularity better than $C^{1,\alpha}_{\text{loc}}$. Hence, I. Holopainen raised the question that it is unclear whether the Bochner's formula has any use at all for $p \neq 2$.

When $p = 2$, $p$-harmonic functions are simply harmonic functions. Liouville type properties or topological end properties have been studied by a long list of authors. We refer the reader to, for example Li [L], Li-Tam [14][15] and Li-Wang [16][17] for further references. In particular, P. Li and J. Wang showed Liouville type properties and splitting type properties on complete noncompact manifolds with positive spectrum $\lambda$ when the Ricci curvature has a lower bound depending on $\lambda$. They also extended their work to a complete noncompact manifold with weighted Poincaré inequality $(P_\mu)$. 
For $p > 1$, we refer the reader to works such as [1][3][7][8][9][10][13][11] for further details. In particular, Kotschwar-Ni [11] (or see Theorem 5) use a Bochner's formula on a neighborhood of the maximum point (i.e. the $p$-Laplace operator is neither degenerate nor singular elliptic on this neighborhood) to prove a gradient estimate for positive $p$-harmonic functions. This also implies Liouville type properties of positive $p$-harmonic functions on complete noncompact manifolds with nonnegative Ricci curvature, and sectional curvature bounded below.

The paper is organized as follows. In section 2, we show the Laplacian comparison theorem and Hessian Comparison Theorem. In section 3, we state Kotschwar-Ni gradient estimate of positive $p$-harmonic functions and extend their result to more general equation associated to the $p$-Laplace operator.
2 Comparison Theorem

In the lecture note of Jiaping Wang [22] (or see [2]) , the following two comparison theorems, Laplacian and Hessian, will be used in this paper, and we write down here.

2.1 Laplacian Comparison Theorem

Let $M$ be a complete $m$-dimensional manifold. Suppose $p$ is a fixed point in $M$, let us consider the distance function $r_p(x) = r(p, x)$ to $p$. When there is no ambiguity, the subscript will be deleted and we will simply write $r(x)$. The distance function in general is not smooth due to the presence of cut-points. However, it can be seen that it is a Lipschitz function with Lipschitz constant 1. In particular, we have

$$|\nabla r|^2 = 1$$

almost everywhere on $M$. Though $r$ might not be a $C^2$ function, one can still estimate its Laplacian in the sense of distribution.

For any unit vector $V$ in the unit tangent sphere $S^{m-1}_p(M)$, we define

$$\rho(V) = \sup\{T : \gamma_V(t) = exp_p(tV) \text{ is minimizing on } [0, T]\}$$

to be the maximum distance for the geodesic in the direction of $V$ to be minimizing. We also let

$$C_p = \{\rho(V)V : \rho(V) < \infty, V \in S^{m-1}_p(M)\}$$

to be the tangential cut locus of $p$. The cut locus of $p \in M$ is denoted by $Cut(p) = exp_p(C_p)$.

Moreover,

$$M = exp_p(\sum(p)) \cup Cut(p),$$

where

$$\sum(p) = \{tV : 0 \leq t < \rho(V), V \in S^{m-1}_p(M)\}$$
and

\[ \exp_p : \Sigma(p) \to \exp_p(\Sigma(p)) \]

is a diffeomorphism. It is known that the set \( \text{Cut}(p) \) has measure zero in \( M \). The polar coordinate system \( (r, \theta) \) on the tangent space \( T_p(M) \) also induces a coordinate chart on \( \exp_p(\Sigma(p)) \). The definition of exponential map implies that \( r(x) = t \) if \( x = \exp_p(t\theta) \) for \( t < \rho(\theta) \). Moreover, \( r(x) \) is smooth on \( \exp_p(\Sigma(p)) \backslash \{p\} \) and \( |\nabla r| = 1 \) on \( \exp_p(\Sigma(p)) \backslash \{p\} \).

We begin by defining the following notion of curvature.

**Definition 1** For any integer \( 1 \leq l \leq m - 1 \), we defined the \( l \)-sectional curvature of a pair \( \{\omega, V\} \), where \( \omega \in T_p M \) and \( V \subset T_p M \) is an \( l \)-dimensional subspace perpendicular to \( \omega \), by

\[ K^l_M(\omega, V) = \sum_{i=1}^{l} (R_{\omega e_i} \omega, e_i) \]

with \( \{e_1, e_2, ..., e_l\} \) being an orthonormal basis for \( V \).

To set up our model for the comparison theorem, we consider \( M^{l+1}_K \) to be the \( (l+1) \)-dimensional, simply connected space form of constant sectional curvature \( K \). For a \( \bar{p} \in M^{l+1}_K \), we denote the distance function from any point \( \bar{x} \) to \( \bar{p} \) by \( \bar{r}(\bar{x}) \).

**Theorem 2** Let \( M \) be a complete Riemannian manifold of dimension \( m \). Assume that the \( l \)-sectional curvatures of \( M \) satisfy \( K^l_m \geq lK \). Then within the cut locus of a fixed point \( p \in M \) and for any \( V \subset T_p M \) perpendicular to \( \nabla r(x) \),

\[ \sum_{i=1}^{l} D^2(r)(e_i, e_i) \leq \begin{cases} 
\frac{l}{t} \sqrt{K \cot(\sqrt{K} t)} & \text{, if } K > 0 \\
\frac{l}{t} & \text{, if } K = 0 \\
\frac{l}{t} \sqrt{|K| \coth(\sqrt{|K|} t)} & \text{, if } K < 0 
\end{cases} \]

with \( \{e_1, ..., e_l\} \) being any orthonormal basis of \( V \) and \( \{\bar{e}_1, ..., \bar{e}_l\} \) being an orthonormal basis of \( T_{\bar{p}} M^{l+1}_K \) with \( \bar{e}_i \perp \nabla r \)
Proof.

For \( x \in exp_p(\sum(p)) \setminus \{p\} \), let \( \gamma \) be the minimal normal geodesic joining \( p \) to \( x \). At \( x \), we choose an orthonormal frame \( \{e_1, ..., e_m\} \), such that \( e_1 = \nabla r \). By parallel translating the frame \( \{e_i\} \) we obtain an orthonormal frame along \( \gamma \) also denoted by \( \{e_i\}_{i=1}^m \) with the property that \( e_1 = \nabla r \). Since \( |\nabla r|^2 = 1 \) on \( exp_p(\sum(p)) \setminus \{p\} \), by taking covariant derivative of this equation, we obtain

\[
0 = (|\nabla r|^2)_{\alpha\alpha} = 2\sum_{i=1}^m r_{i\alpha}r_{i\alpha} + 2\sum_{i=1}^m r_i r_{i\alpha\alpha}
\]

for each \( 2 \leq \alpha \leq m \). Since \( \gamma \) is a geodesic and each \( e_i \) is parallel along \( \gamma \), each term on the right hand side of (2) can be interpreted as covariant derivatives. The commutation formula for covariant derivative then implies

\[
\sum_{i=1}^m r_i r_{i\alpha\alpha} = \sum_{i=1}^m r_{i\alpha}r_{\alpha i} + \sum_{i,j=1}^m R_{i\alpha j\beta} r_{i\beta}.
\]

Substituting into (2) and using the fact that \( |\nabla r| = 1 = r_1 \), we obtain

\[
0 \geq 2r_{\alpha\alpha}^2 + 2\frac{\partial (r_{\alpha\alpha})}{\partial r} + 2K_M(e_1,e_\alpha).
\]

Suppose \( V \subset T_xM \) is spanned by \( \{e_2, ..., e_{l+1}\} \), then summing over \( \alpha = 2, ..., l + 1 \), (3) becomes

\[
0 \geq \sum_{\alpha=2}^{l+1} r_{\alpha\alpha}^2 + \frac{\partial}{\partial r} \left( \sum_{\alpha=2}^{l+1} r_{\alpha\alpha} \right) + K_M(e_1,V).
\]

Using the lower bound of the \( l \)-sectional curvature, the inequality

\[
\sum_{\alpha=2}^{l+1} r_{\alpha\alpha}^2 \geq \frac{1}{l} \left( \sum_{\alpha=2}^{l+1} r_{\alpha\alpha} \right)^2,
\]

and by setting \( f(t) = \sum_{\alpha=2}^{l+1} r_{\alpha\alpha}(\gamma(t)) \), (4) can be expressed as

\[
0 \geq \frac{1}{l} f^2(t) + f'(t) + lK.
\]
Note that since a smooth Riemannian metric is locally Euclidean,

\[ \lim_{t \to 0} t f(t) = l. \]

We will now consider the three separate cases when \( K = 0 \), \( K > 0 \), and \( K < 0 \).

Case 1: When \( K = 0 \), inequality (5) becomes

\[ \frac{1}{t} f'(t) + f'(t) \leq 0. \]

This implies that \( f'(t) \leq 0 \) and \( f(t) \) is a decreasing function. Let \((0, T)\) be the largest interval such that \( f(t) > 0 \), then we have

\[ (\frac{1}{t})' = -\frac{f'}{f^2} \geq \frac{1}{t} \]

and \( f(t) \leq \frac{t}{l} \) on \((0, T)\). Since \( f(t) \leq 0 \) for \( t \geq T \), we can still conclude that \( f(t) \leq \frac{t}{l} \) on \((0, \rho(\theta))\).

Case 2: When \( K > 0 \), inequality (5) can be written as

\[ \frac{f'(t)}{f^2(t) + t^2 K} \leq -1. \]

This implies that

\[ \frac{d}{dt} \tan^{-1}(\frac{f}{t\sqrt{K}}) \leq -\sqrt{K}. \]

Integrating from 0 to \( t \), we have

\[ \tan^{-1}(\frac{f}{t\sqrt{K}}) \leq \frac{\pi}{2} - \sqrt{K} t, \]

implying that

\[ f(t) \leq l\sqrt{K} \cot(\sqrt{K} t). \]

Case 3: When \( K < 0 \), let \( T \) be the first time such that

\[ f^2(t) + t^2 K = 0, \]
Then on \((0, T)\), we have \(f^2(t) + l^2 K > 0\) and
\[
\frac{f'(t)}{f^2(t) + l^2 K} \leq -1.
\]

This implies that
\[
\frac{d}{dt} \coth^{-1}\left(\frac{f}{\sqrt{|K|}}\right) \geq \sqrt{|K|},
\]
and
\[
f(t) \leq l\sqrt{|K|} \coth(\sqrt{|K|} t)
\]
on \((0, T)\). For \(t \geq T\) we claim that \(f(t) \leq l\sqrt{|K|}\). Indeed, if \(f(t_1) > l\sqrt{|K|}\) for \(t_1 > T\), then there exists \(t_2 \in (T, t_1)\) such that \(f'(t_2) \geq 0\) and \(f(t_2) > l\sqrt{|K|}\). In this case,
\[
f'(t_2) + \frac{1}{t} f^2(t_2) + lK > 0,
\]
which is a contradiction.

Thus,
\[
f(t) \leq l\sqrt{|K|}
\]
for \(T < t < \rho(\theta)\), and we conclude that
\[
f(t) \leq l\sqrt{|K|} \coth(\sqrt{|K|} t)
\]
for \(0 < t < \rho(\theta)\).

\[\Box\]

### 2.2 Hessian Comparison Theorem

The proof of Theorem 2 also implies the following two types of Hessian comparison theorems whenever \((M^n, g_2)\) is space form.

Note that the mean curvature \(H\) of \(S(p, r)\) is \(H = (n - 1)\frac{\omega}{\sigma}\). In particular for the constant
curvature $K$, the mean curvature $H_K(r)$ of the distance sphere $S_R(p, r)$ is

$$H_K(r) \triangleq \begin{cases} (n-1)\sqrt{K} \cot(\sqrt{K}t) & \text{if } K > 0 \\ \frac{n-1}{t} & \text{if } K = 0 \\ (n-1)\sqrt{|K|} \coth(\sqrt{|K|}t) & \text{if } K < 0 \end{cases}$$

**Theorem 3** Let $i = 1, 2$. Let $(M^n_i, g_i)$ be complete Riemannian $n$-manifold, let $\gamma_i : [0, L] \to M^n_i$ be geodesics parametrized by arc length such that $\gamma_i$ does not intersect the cut locus of $\gamma_i(0)$, and let $d_i \triangleq d(\cdot, \gamma_i(0))$. If for all $t \in [0, L]$ we have

$$K_{g_1}(V_1 \wedge \dot{\gamma}_1(t)) \geq K_{g_2}(V_2 \wedge \dot{\gamma}_2(t))$$

for all unit vectors $V_i \in T_{\gamma_i}M^n_i$ perpendicular to $\dot{\gamma}_i(t)$, then

$$\nabla \nabla d_1(X_1, X_1) \leq \nabla \nabla d_2(X_2, X_2)$$

for all $X_i \in T_{\gamma_i(0)}M^n_i$ perpendicular to $\dot{\gamma}_i(t)$ and $t \in (0, L]$.

**Theorem 4** Let $(M^n, g)$ be a complete Riemannian manifold with $K_M \geq K$. For any point $p \in M$ the distance function $r(x) \triangleq d(x, p)$ satisfies

$$\nabla_i \nabla_j r = h_{ij} \leq \frac{1}{n-1} H_K(r) g_{ij}$$

at all points where $r$ is smooth. On all of $M$ the above inequality holds in the sense of support functions.
3 Gradient estimate for $p$-Laplace’s equations

3.1 The work of B. Kotschwar and L. Ni

In 2009, B. Kotschwar and L. Ni proved the gradient estimate for $p$-Laplace’s equation in [11]. We state and prove their result as follows.

Let $v$ be a positive $p$-harmonic function, i.e., a function satisfying

$$\text{div}(|\nabla v|^{p-2}\nabla v) = 0.$$ 

Denote $u \triangleq -(p - 1) \log v$. It is easy to see that $u$ satisfies

$$\text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p.$$ 

**Theorem 5** Assume that $v$ is a positive $p$-harmonic function on the ball $B(x_0, R)$ the sectional curvature of $(M, g)$, $K_M \geq -K^2$. Then for any $\varepsilon > 0$,

$$\sup_{B(x_0, \frac{R}{2})} |\nabla u|^2 \leq \frac{20(n-1)}{R^2(1-\varepsilon)} \left( c_{p,n} + \frac{(n-1)K^2}{8\varepsilon} \right) + C(n, K, p, R, \varepsilon)$$

where

$$b_{p,n} = \frac{2(p-1)}{n-1} - 2,$$

$$c_{p,n} = \left( \frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2} \right)_- + \left( \left( \frac{p}{2} + 1 \right) \max\{p - 1, 1\} \right),$$

and

$$C(n, K, p, R, \varepsilon) = \frac{(n-1)^2}{1-\varepsilon} K^2 + \frac{40(n+p-1)(n-1)K^2}{1-\varepsilon} + \frac{20\max\{p-1,1\}(n-1)}{(1-\varepsilon)R^2}.$$ 

If $v$ is defined globally, by taking $R \to \infty$, and $\varepsilon \to 0$, the above theorem implies that

$$|\nabla u|^2 \leq (n-1)^2 K^2.$$ 

**Proof.** Let $u, f$ be as above. Assume that $f > 0$ over some region of $M$. 

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Define
\[ L(\psi) \doteq \text{div}[f^{p/2-1}A(\nabla \psi)] - pf^{p/2-1}(\nabla u, \nabla \psi), \]
here
\[ A = id + (p - 2)\frac{\nabla u \otimes \nabla u}{f} \]
which can be checked easily to be nonnegative definite in general and positive definite for \( p > 1 \). Note that the operator \( L \) is the linearized operator of the nonlinear equation
\[ \text{div}(|\nabla u|^{p-2}\nabla u) = |\nabla u|^p. \]
The first is a computational lemma.

Lemma 6

(7) \[ L(f) = 2f^{p/2-1}(u_{ij}^2 + R_{ij}u_iu_j) + (\frac{p}{2} - 1)|\nabla f|^2f^{p/2-2}. \]
here \( u_{ij} \) is the Hessian of \( u \). \( R_{ij} \) is the Ricci curvature of \( M \).

**Proof.** Direct calculation shows that
\[
L(f) = (\frac{p}{2} - 1)f^{p/2-2}|\nabla f|^2 + f^{p/2-1}\Delta f + (p - 2)\Delta u(\nabla u, \nabla f)f^{p/2-2} \\
+ (p - 2)(\frac{p}{2} - 1)f^{p/2-3}(\nabla u, \nabla f)^2 \\
+ (p - 2)(u_{ij}f_{ij}f^{p/2-2} + f_{ij}u_iu_jf^{p/2-2} - (\nabla u, \nabla f)^2f^{p/2-3}) \\
- pf^{p/2-1}(\nabla u, \nabla f).
\]
Using
\[ \Delta f = 2u_{ij}^2 + 2(\nabla \Delta u, \nabla u) + 2R_{ij}u_iu_j \]
and combining terms we have that
\[
L(f) = f^{p/2-1}(2u_{ij}^2 + 2(\nabla \Delta u, \nabla u) + 2R_{ij}u_iu_j) + (\frac{p}{2} - 1)f^{p/2-2}|\nabla f|^2 \\
+ (p - 2)\Delta u(\nabla u, \nabla f)f^{p/2-2} + (p - 2)(\frac{p}{2} - 2)f^{p/2-3}(\nabla u, \nabla f)^2 \\
+ (p - 2)(u_{ij}f_{ij}f^{p/2-2} + f_{ij}u_iu_jf^{p/2-2}) - pf^{p/2-1}(\nabla u, \nabla f).\]
Taking the gradient of both sides of
\begin{equation}
\left(\frac{p}{2} - 1\right) f^{p/2 - 2} \langle \nabla f, \nabla u \rangle + f^{p/2 - 1} \Delta u = f^{p/2},
\end{equation}
and computing its product with $\nabla u$, we have that
\begin{align*}
\left(\frac{p}{2} - 1\right) \left(\frac{p}{2} - 2\right) f^{p/2 - 3} \langle \nabla f, \nabla u \rangle^2 + \left(\frac{p}{2} - 1\right) f^{p/2 - 2} (f_{ij} u_i u_j + u_i f_i u_j) \\
+ \left(\frac{p}{2} - 1\right) f^{p/2 - 2} \Delta u \langle \nabla f, \nabla u \rangle + f^{p/2 - 1} \langle \nabla \Delta u, \nabla u \rangle = \frac{p}{2} f^{p/2 - 1} \langle \nabla f, \nabla u \rangle.
\end{align*}
Combining the above two equalities, we prove the claimed identity (7).

Now let $\eta(x) = \theta(\frac{r(x)}{R})$, where $\theta(t)$ is a cut-off function such that $\theta(t) \equiv 1$ for $0 \leq t \leq \frac{1}{2}$ and $\theta(t) \equiv 0$ for $t \geq 1$. Furthermore, take the derivatives of $\theta$ to satisfy $\frac{(\theta')^2}{\theta} \leq 10$ and $\theta'' \geq -10 \theta \geq -10$. Here $r(x)$ denotes the distance from some fixed $x_0$. Let $Q = \eta f$, which vanishes outside $B(x_0, R)$. At the maximum point of $Q$, it is easy to see that
\begin{equation}
\nabla Q = (\nabla \eta) f + (\nabla f) \eta = 0
\end{equation}
and
\begin{equation*}
0 \geq L(Q).
\end{equation*}
On the other hand, at the maximum point,
\begin{align*}
L(Q) &= \eta \text{div} \left( f^{p/2 - 1} A \nabla f \right) - \eta f^{p/2 - 1} \langle \nabla f, \nabla f \rangle \\
&\quad + f^{p/2 - 1} \langle A(\nabla f), \nabla \eta \rangle + \text{div} \left( f^{p/2} A \nabla \eta \right) - p f^{p/2} \langle \nabla u, \nabla \eta \rangle \\
&= \eta L(f) - \left(\frac{p}{2} + 1\right) f^{p/2} \frac{\langle A(\nabla \eta), \nabla \eta \rangle}{\eta} + f^{p/2} \text{div} \left( A \nabla \eta \right) - p f^{p/2} \frac{\langle \nabla u, \nabla \eta \rangle}{\eta}.
\end{align*}
If $1 < p \leq 2$,
\begin{equation*}
\frac{\langle A(\nabla \eta), \nabla \eta \rangle}{\eta} \leq \frac{|\nabla \eta|^2}{\eta}.
\end{equation*}
For $p \geq 2$ we have that
\begin{equation*}
\frac{\langle A(\nabla \eta), \nabla \eta \rangle}{\eta} \leq (p - 1) \frac{|\nabla \eta|^2}{\eta}.
\end{equation*}
The next lemma estimates $\text{div} \left( A \nabla \eta \right)$. 
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Lemma 7 Assume that, on $B(x_0, R)$, the sectional curvature of $(M, g)$ satisfies $K_M \geq -K^2$.
Then, at the maximum point of $Q$,

\begin{equation}
\text{div}(A \nabla \eta) \geq -20(n + p - 2) \frac{1 + \frac{K}{R}}{R^2} - \frac{10 \max\{p-1,1\}}{R^2} + (p - 2) \langle \nabla u, \nabla \eta \rangle \\
+ (p - 2) \frac{\eta}{\eta_f} (\frac{\nabla u, \nabla \eta}{\eta_f})^2 - (\frac{p}{2} - 1) \frac{\eta^2}{\eta}.
\end{equation}

**Proof.** Direct computation yields

\[
\text{div}(A \nabla \eta) = \Delta \eta + \frac{(p-2) u_i u_j}{f} + (p - 2) \Delta u \frac{(\nabla u, \nabla \eta)}{f} - \frac{(\nabla u, \nabla f) (\nabla u, \nabla \eta)}{f^2} + (p - 2) \frac{u_i u_j \eta}{f}.
\]

Now using (9), (8) and that $f_j = 2u_i u_j u_i$, we can eliminate $\Delta u$ and $\nabla f$ to arrive at

\[
\text{div}(A \nabla \eta) = \Delta \eta + (p - 2) \frac{\eta_i u_i u_j}{f} + (p - 2) \langle \nabla u, \nabla \eta \rangle \\
+ (p - 2) \frac{\eta^2}{\eta_f} - \frac{\eta^2}{\eta_f}.
\]

We only need to estimate the first two terms, for which we compute

\[
\eta_{ij} = \theta \frac{r_i r_j}{R} + \theta^n \frac{r_i r_j}{R^2}.
\]

Using the Hessian comparison theorem [6], which states that $r_{ij} \leq \frac{1 + \frac{K}{r}}{r} g_{ij}$, and the Laplacian comparison theorem, we have that

\[
A_{ij} r_{ij} \leq (n + p - 2) \frac{1 + \frac{K}{r}}{r}.
\]

Noting that $\theta t = 0$ if $r \leq \frac{R}{2}$, we have that

\[
\Delta \eta + (p - 2) \frac{\eta_i u_i u_j}{f} = A_{ij} \eta_{ij} \geq -20(n + p - 2) \frac{1 + \frac{K}{R}}{R^2} - 10 \max\{p-1,1\} \frac{1}{R^2}.
\]

Taken together, these estimates prove the lemma. \blacksquare

To prove the theorem 5, we first estimate $L(f)$ from below. We only need to estimate it over the points where $f > 0$ for our purpose of estimating $f$ from above. Choose a local
orthonormal frame \{e_i\} near any such given point so that at the given point \( \nabla u = |\nabla u|e_1 \). Then \( f_1 = 2u_{j1}u_j = 2u_{11}u_1 \) and for \( j \geq 2 \), \( f_j = 2u_{j1}u_1 \), which imply that

\[
2u_{k1} = \frac{f_k}{f^{1/2}}.
\]

Now (8) becomes

\[
\sum_{j \geq 2} u_{jj} = f - \left( \frac{p}{2} - 1 \right) \frac{f_{11}}{f} - u_{11}.
\]

Hence

\[
\sum_{i,j=1}^n u_{ij}^2 \geq u_{11}^2 + 2 \sum_{j \geq 2} u_{j1}^2 + \sum_{j \geq 2} u_{jj}^2
\]

\[
\geq u_{11}^2 + 2 \sum_{j \geq 2} u_{j1}^2 + 1n - 1(Xj \geq 2ujj) 2
\]

\[
= \frac{n-1}{n-1} u_{11}^2 + 2 \sum_{j \geq 2} u_{j1}^2 + \frac{1}{n-1} f^2 + \frac{1}{n-1} \left( \frac{p}{2} - 1 \right)^2 \frac{(f_{11})^2}{f^2}
\]

\[- \frac{2}{n-1} \left( \frac{p}{2} - 1 \right) f_{11} u_{11} - \frac{2}{n-1} f_{11} u_{11} + \frac{2}{n-1} \left( \frac{p}{2} - 1 \right) f_{11} u_{11} u_{11}.
\]

Using (11), we can replace all the second derivatives of \( u \) and arrive at

\[
\sum_{i,j=1}^n u_{ij}^2 \geq \frac{1}{n-1} f^2 + \frac{1}{n-1} \left( \frac{n}{4} + \frac{p}{2} - 1 \right) \frac{f_{11}^2}{f} + \frac{1}{n-1} \sum_{j \geq 2} \frac{f_{jj}^2}{f}
\]

\[
+ \frac{1}{n-1} \left( \frac{p}{2} - 1 \right)^2 \frac{(\nabla f \cdot \nabla u)^2}{f^2} - \frac{2}{n-1} \frac{(\nabla f \cdot \nabla u)^2}{f^2} - \frac{n-1}{n-1} \frac{(\nabla f \cdot \nabla u)^2}{f^2} - \frac{p-1}{n-1} \frac{(\nabla f \cdot \nabla u)^2}{f^2}
\]

where

\[
a_{n,p} \triangleq \min \left\{ \frac{1}{n-1} \left( \frac{n}{4} + \frac{p}{2} - 1 \right), \frac{1}{2} \right\} \geq 0.
\]

Hence by (7), (9) we have that

\[
f^{p/2-1} L(f) \geq \frac{2}{n-1} f^p + 2 a_{n,p} f^{p-3} |\nabla f|^2 + \frac{2}{n-1} f^{p-2} \left( \frac{\nabla u \cdot \nabla u}{\eta} \right)^2
\]

\[
+ \frac{2}{n-1} f^{p-1} \left( \frac{\nabla u \cdot \nabla \eta}{\eta} \right)^2 - 2(n-1) K^2 f^{p-1} + \left( \frac{p}{2} - 1 \right) f^{p-1} \left( \frac{\nabla \eta}{\eta} \right)^2.
\]

Now combining the previous estimates, we have that

\[
0 \geq f^{p/2-1} \eta^{p-1} L(Q)
\]

\[
\geq \frac{2}{n-1} Q^p + Q^{p-2} \left[ \frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2} \right] \frac{(\nabla u \cdot \nabla \eta)^2}{\eta^2}
\]

\[
+ \left[ \frac{2(p-1)}{n-1} - 2 \right] Q^{p-1} (\nabla u \cdot \nabla \eta) - \left[ \frac{p}{2} + 1 \right] \max \{ p - 1, 1 \} \frac{Q^{p-1} |\nabla \eta|^2}{\eta^2}
\]

\[- \left[ 2(n-1) K^2 + 20(n+p-3) \frac{1+KR}{R^2} + \frac{10 \max\{p-1,1\}}{R^{p-1}} \right] Q^{p-1}.
\]
Since
\[ Q^{p-2} \left[ \frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2} \right] (\nabla u, \nabla \eta)^2 \geq -\frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2} \right]_+ \geq Q^{p-1} \frac{\nabla \eta^2}{\eta} \]
and
\[ \frac{2(p-1)}{n-1} - 2 \right] Q^{p-1} \left( \nabla u, \nabla \eta \right) \geq -\frac{2}{n-1} Q^p - \frac{b_{p,n}^2 (n-1)}{8 \varepsilon} Q^{p-1} \frac{\nabla \eta^2}{n} \]
with \( b_{p,n} = \frac{2(p-1)}{n-1} - 2 \), we have that
\[
0 \geq f^{\eta/2-1} \eta^{p-1} L(Q) \\
\geq \frac{2(1-c)}{n-1} Q^p - \frac{10}{R^2} Q^{p-1} \left[ c_{p,n} + \frac{b_{p,n}^2 (n-1)}{8 \varepsilon} \right] \\
- \left[ 2(n-1)K^2 + 20(n + p - 3) \frac{1 + KR}{R^2} + \frac{10 \max \{ p-1, 1 \}}{R^2} \right] \frac{Q^{p-1}}{Q} \]
where
\[
c_{p,n} = \frac{2(p/2-1)^2}{n-1} + \frac{(p-2)p}{2} + \left[ \frac{p}{2} + 1 \right] \max \{ p - 1, 1 \}, \]
here we have used \( \frac{\nabla \eta^2}{\eta} \leq \frac{10}{R^2} \).

Theorem 5 then follows easily from the above inequality. \( \Box \)
3.2 Gradient estimate for \( \Delta_p v = -\lambda_1 |v|^{p-2} v - \lambda |\nabla v|^{p-2} v \)

Now we consider the positive \( C^3(M) \) solution of the equation associated to the \( p \)-Laplace operator

\[
(14) \quad \Delta_p v = -\lambda_1 |v|^{p-2} v - \lambda |\nabla v|^{p-2} v, \quad p > 1.
\]

As the method of P. Li [L], B. Kotschwar and L. Ni [11], let \( u = -(p-1) \log v \), then

**Lemma 8** Assume \( v \) is a positive \( C^3(M) \) solution of \((14)\). Let \( u = -(p-1) \log v \), then

\[
(15) \quad \nabla u = \frac{-(p-1) \nabla v}{v},
\]

and

\[
(16) \quad \Delta_p u = \lambda_1 (p-1)^{p-1} + \lambda (p-1) |\nabla u|^{p-2} + |\nabla u|^p
\]

whenever \( \nabla u \neq 0 \).

**Proof.** It is easy to check \((15)\) holds. According to \((15)\), we have

\[
\Delta_p u = \text{div}(|\nabla u|^{p-2} \cdot \nabla u)
\]

\[
= \text{div}((p-1)^{p-2} \cdot \left[|\nabla u|^{p-2} \cdot \frac{-(p-1) \nabla v}{v}\right])
\]

\[
= -(p-1)^{p-1} \text{div}\left[|\nabla u|^{p-2} \cdot \frac{-\lambda |v|^{p-2} v - \lambda |\nabla v|^{p-2} v}{v}ight]
\]

\[
= -(p-1)^{p-1} \left[\text{div}(\frac{\lambda}{v}) - \frac{(p-1) |\nabla v|^p}{v}\right]
\]

\[
= \lambda_1 (p-1)^{p-1} + \lambda (p-1) \cdot \left|\frac{(p-1) \nabla v}{v}\right|^{p-2} + \left|\frac{(p-1) \nabla v}{v}\right|^p
\]

Then Lemma 8 follows. \(\blacksquare\)

Now let

\[
f = |\nabla u|^2 = (p-1)^2 |\nabla v|^2,
\]

and assume \( f > 0 \) over some region of \( M \).
Lemma 9

(17) \((\frac{p}{2} - 1)f^{\frac{p}{2} - 2}(\nabla u, \nabla f) + f^{\frac{p}{2} - 1}\Delta u = \lambda_1(p-1)^{p-1} + \lambda(p-1)f^{\frac{p-2}{2}} + f^\frac{p}{2}\)

whenever \(f \neq 0\) at \(x \in M\).

Proof. By using

\[
\Delta_{\rho} u = \text{div}(\nabla u) \\
= \text{div}(f^{\frac{p}{2} - 1}\nabla u) \\
= (\frac{p}{2} - 1)f^{\frac{p}{2} - 2}(\nabla u, \nabla f) + f^{\frac{p}{2} - 1}\Delta u,
\]

and (16), then we obtain

\[(\frac{p}{2} - 1)f^{\frac{p}{2} - 2}(\nabla u, \nabla f) + f^{\frac{p}{2} - 1}\Delta u = \lambda_1(p-1)^{p-1} + \lambda(p-1)f^{\frac{p-2}{2}} + f^\frac{p}{2}\]

which is (17). \(\blacksquare\)

Take \(\eta(x) = \theta(\frac{r(x)}{R})\), where \(r(x)\) is the distance function from the fixed point \(x_0\) and \(\theta(t)\) is a cut-off function satisfying

\[
\begin{cases}
\theta (t) = 1 & \text{if } 0 \leq t \leq 1/2, \\
0 \leq \theta (t) \leq 1 & \text{if } 1/2 \leq t \leq 1, \\
\theta (t) = 0 & \text{if } t \geq 1.
\end{cases}
\]

and \(\frac{\theta'^2}{\theta'} \leq 10, \theta'' \geq -10\theta \geq -10\). Hence, \(Q = \eta \cdot f\) vanishes outside \(B(x_0, R)\).

Now assume \(Q\) has a maximum at \(p_0 \in B(x_0, R)\). We may assume \(f(p_0) \neq 0\) since \(f(p_0) = 0\) infers that \(u\) is constant on \(B(x_0, R)\). Moreover, we have \(\nabla Q = 0\) and \(\nabla^2 Q \leq 0\) at this point \(p_0\). So we have

\[
\nabla Q = (\nabla \eta)f + (\nabla f)\eta = 0
\]

which is

(18) \(\nabla f = -\frac{f\nabla \eta}{\eta}\)

at \(p_0\).
Definition 10 Let

\[ \mathcal{L}(\Psi) = \text{div}[f^{\frac{p}{2} - 1} \cdot A(\nabla \Psi)] \]

be the linearized operator of the Laplace operator, where

\[ A(\cdot) = \text{Id} + (p - 2)f^{-1}(\nabla u, \cdot)\nabla u. \]

Proposition 11 \( \mathcal{L}(Q) \leq 0 \) at \( p_0 \).

Proof. Since \( \nabla Q = 0 \) at \( p_0 \), we obtain

\[
\mathcal{L}(Q) = \text{div}[f^{\frac{p}{2} - 1}(\nabla Q + (p - 2)f^{-1}(\nabla u, \nabla Q)\nabla u)] \\
= \left( \frac{p}{2} - 1 \right)f^{\frac{p}{2} - 2}(\nabla f, [\nabla Q + (p - 2)f^{-1}(\nabla u, \nabla Q)\nabla u]) \\
+ f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)\text{div}(f^{-1}(\nabla u, \nabla Q)\nabla u)] \\
= 0 + f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)\text{div}(f^{-1}(\nabla u, \nabla Q)\nabla u)] \\
+ (p - 2)f^{-1}\nabla ^2 u(\nabla u, \nabla Q) + (p - 2)f^{-1}\nabla ^2 Q(\nabla u, \nabla u) \\
= f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)f^{-1}\nabla ^2 Q(\nabla u, \nabla u)] \\
= f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)\nabla ^2 Q(\tilde{e}, \tilde{e})]
\]

at \( p_0 \), where \( \tilde{e} = f^{\frac{1}{2}}\nabla u \). It is clear that \( \nabla ^2 Q(\tilde{e}, \tilde{e}) \leq 0 \) and \( \Delta Q \leq 0 \) at \( p_0 \) since \( \nabla ^2 Q \leq 0 \) at \( p_0 \).

If \( p \geq 2 \), then

\[ \mathcal{L}(Q) = f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)\nabla ^2 Q(\tilde{e}, \tilde{e})] \leq 0. \]

If \( 1 < p < 2 \), we also have

\[ \mathcal{L}(Q) = f^{\frac{p}{2} - 1}[\Delta Q + (p - 2)\nabla ^2 Q(\tilde{e}, \tilde{e})] \leq 0, \]

since, let \( \{\lambda_i\} \) be the non-positive eigenvalues of \( \nabla ^2 Q \) at \( p_0 \), with

\[ \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 0, \]
then
\[
\mathcal{L}(Q) = f^{\frac{p}{p-1}}[\Delta Q + (p-2)\nabla^2 Q(\bar{c}, \bar{c})]
\leq f^{\frac{p}{p-1}}[\sum_{i=1}^{n} \lambda_i + (p-2)\lambda_1]
= f^{\frac{p}{p-1}}\left(\sum_{i=2}^{n} \lambda_i + (p-1)\lambda_1\right)
\leq 0.
\]

Thus \(\mathcal{L}(Q) \leq 0\) at \(p_0\). \(\blacksquare\)

\textbf{Lemma 12} We use the linearized operator \(\mathcal{L}\) to calculate the \(\mathcal{L}(f)\) as follows:
\[
\mathcal{L}(f) = (\frac{\nu}{2} - 1)f^{\frac{\nu}{\nu-2}}|\nabla \eta|^2 + 2f^{\frac{\nu}{\nu-1}}(u_{ij}^2 + R_{ij}u_iu_j)
-pf^{\frac{\nu}{\nu-1}}(\nabla f, \nabla \eta) - \lambda(p-1)^{p-1}(p-2)f^{\frac{\nu}{\nu-1}}\eta^{-1}(\nabla u, \nabla \eta),
\]
at \(p_0\), here \(u_{ij}\) is the Hessian of \(u\), \(R_{ij}\) is the Ricci curvature of \(M\).

\textbf{Proof.} Direct calculation shows that
\[
\mathcal{L}(f) = \text{div}[f^{\frac{\nu}{\nu-1}}(\nabla f + (p-2)f^{-1}(\nabla u, \nabla f)\nabla u)]
= \frac{\nu-2}{2}f^{\frac{\nu}{\nu-2}}|\nabla f|^2 + \frac{(p-2)^2}{2}f^{\frac{\nu}{\nu-3}}(\nabla u, \nabla f)^2
+f^{\frac{\nu}{\nu-1}}\Delta f + (p-2)f^{\frac{\nu}{\nu-1}}(\nabla (f^{-1}(\nabla u, \nabla f)))\nabla u
+(p-2)f^{\frac{\nu}{\nu-2}}(\nabla u, \nabla f)\Delta u.
\]

Taking the gradient of \(\Delta u\) of lemma 9 and computing its inner product with \(\nabla u\), we have that
\[
(19) \quad \langle \nabla \Delta u, \nabla u \rangle = -\frac{\lambda_1(p-1)^{p-1}(p-2)}{2f^{\frac{\nu}{\nu}}} \langle \nabla f, \nabla u \rangle + \langle \nabla f, \nabla u \rangle - \frac{p-2}{2} \left( \nabla \left( \frac{\nabla u}{f^\nu} \right) \right) \nabla u.
\]

Then Bochner’s formula
\[
\Delta f = 2u_{ij}^2 + 2(\nabla \Delta u, \nabla u) + 2R_{ij}u_iu_j,
\]
19
(17) and (19) give
\[
\mathcal{L}(f) = \frac{\nu-2}{2} f^{\frac{\nu}{2}-2} |\nabla f|^2 + \frac{(\nu-2)^2}{2} f^{\frac{\nu}{2}-3} \langle \nabla u, \nabla f \rangle^2 \\
+ f^{\frac{\nu}{2}-1} \left( 2u^2_{ij} + 2 \langle \nabla \Delta u, \nabla u \rangle + 2 R_{ij} u_i u_j \right) \\
+ f^{\frac{\nu}{2}-1} \left( -\lambda_1 (p-1)^{p-1} (p-2) f^{-\frac{\nu}{2}} \langle \nabla f, \nabla u \rangle + 2 \langle \nabla f, \nabla u \rangle - 2 \langle \nabla \Delta u, \nabla u \rangle \right) \\
+ (p-2) f^{\frac{\nu}{2}-2} \langle \nabla u, \nabla f \rangle \left( \frac{\lambda_1 (p-1)^{p-1}}{f^{\frac{\nu}{2}-1}} + (p-1)^{p-1} \lambda + f \frac{(p-1)^{p-1} \langle \nabla u, \nabla f \rangle}{f} \right) \\
= \frac{\nu-2}{2} f^{\frac{\nu}{2}-2} |\nabla f|^2 + f^{\frac{\nu}{2}-1} \left( 2u^2_{ij} + 2 R_{ij} u_i u_j \right) \\
+ \lambda (p-1)^{p-1} (p-2) f^{\frac{\nu}{2}-3} \langle \nabla u, \nabla f \rangle + pf^{\frac{\nu}{2}-1} \langle \nabla u, \nabla f \rangle.
\]

So \( \nabla f = -f \eta^{-1} \langle \nabla \eta \rangle \) implies that the identity becomes
\[
\mathcal{L}(f) = \left( \frac{p}{2} - 1 \right) f^{\frac{\nu}{2}-2} |\nabla \eta|^2 + 2 f^{\frac{\nu}{2}-1} (u^2_{ij} + R_{ij} u_i u_j) \\
- pf^{\frac{\nu}{2}-1} \langle \nabla u, \nabla \eta \rangle - \lambda (p-1)^{p-1} (p-2) f^{\frac{\nu}{2}-1} \eta^{-1} \langle \nabla u, \nabla \eta \rangle
\]
at \( p_0 \). \n
Lemma 13 Assume that, on \( B(x_0, R) \), the section curvature of \( (M, g) \) satisfies \( K_M \geq -K^2 \).

Then, at \( p_0 \),
\[
\text{div}[A(\nabla \eta)] \geq -40 R^{-2} [2(\nu + p - 2)(1 + KR) + b] + \frac{p}{2} (p-2) f^{-1} \eta^{-1} (\langle \nabla u, \nabla \eta \rangle)^2 \\
- (\frac{p}{2} - 1) \eta^{-1} |\nabla \eta|^2 + \lambda_1 (p-2)(p-1)^{p-1} f^{\frac{\nu}{2}} \langle \nabla u, \nabla \eta \rangle \\
+ \lambda (p-1)^{p-1} (p-2) f^{-1} \langle \nabla u, \nabla \eta \rangle + (p-2) \langle \nabla u, \nabla \eta \rangle,
\]
where \( b = \max\{1, p-1\} \).

Proof. Since (17), (18) and
\[
u_{ij} u_j = \frac{1}{2} f_i = \frac{1}{2} f \eta^{-1} \eta_i,
\]
}\]
\[ \text{imply} \]
\[ \text{div}[A(\nabla \eta)] = \text{div}[\nabla \eta + (p - 2)f^{-1}(\nabla u, \nabla \eta)\nabla u] \]
\[ = \Delta \eta - (p - 2)f^{-2}(\nabla u, \nabla \eta)(\nabla u, \nabla f) + (p - 2)f^{-1}(u_{ij} \eta_{i} u_{j} + u_{i} \eta_{i} u_{j}) \]
\[ + (p - 2)f^{-1}(\nabla u, \nabla \eta) \Delta u \]
\[ = \Delta \eta + (p - 2)f^{-1} \eta^{-1}(\nabla u, \nabla \eta)^2 + (p - 2)f^{-1}(\frac{-1}{2}f \eta^{-1} |\nabla \eta|^2 + \eta_{i} u_{i} u_{j}) \]
\[ + \lambda(1 - p)(p - 1)^{p-1}f^{-\frac{p}{2}}(\nabla u, \nabla \eta) + \frac{1}{2}(p - 2)^{(p - 2)f^{-1} \eta^{-1}(\nabla u, \nabla \eta)^2} \]
\[ + (p - 2)(\nabla u, \nabla \eta) + \frac{b}{2} - 1)(p - 2)f^{-1} \eta^{-1}(\nabla u, \nabla \eta)^2 \]
\[ = \Delta \eta + (p - 2)f^{-1}(\eta_{ij} u_{i} u_{j}) + \frac{p(p - 2)(\nabla u, \nabla \eta)^2}{2f \eta} - \frac{p - 2}{2} \eta^{-1} |\nabla \eta|^2 \]
\[ + (p - 2)\left(\lambda(1 - p)^{p-1}f^{-\frac{p}{2}} + \lambda(p - 1)^{p-1}f^{-1} + 1\right) \langle \nabla u, \nabla \eta \rangle, \]

So we only need to compute \( \Delta \eta + (p - 2)f^{-1}(\eta_{ij} u_{i} u_{j}) \).

It is clear that \( \eta = \theta(\frac{r}{R}) \) gives

\[
\begin{cases} 
\eta = \theta' \cdot \frac{r}{R}, \\
\eta_{ij} = \theta'' \cdot \frac{r_{ij}}{R} + \theta' \cdot \frac{r_{ij}}{R}. 
\end{cases}
\]

Let \( b = \max\{1, p - 1\} \). Using the Hessian comparison theorem \( r_{ij} \leq \frac{1 + Kr}{r} \cdot g_{ij} \), and the Laplacian comparison theorem, we have, by \( \frac{(\theta'')^2}{\theta} \leq 10 \) and \( \theta'' \geq -10 \theta' \geq -10 \),

\[
\begin{align*}
\Delta \eta + (p - 2)f^{-1}(\eta_{ij} u_{i} u_{j}) & = \theta'' \cdot \frac{|\nabla r|^2}{R^2} + \theta' \cdot \frac{\Delta r}{R} + (p - 2)\left(\theta'' \cdot \frac{r_{i} r_{j} u_{i} u_{j}}{R^2 f} + \theta' \cdot \frac{r_{ij} u_{i} u_{j}}{R f}\right) \\
& \geq \frac{\theta'}{R} \cdot \left( n + p - 2 \right) \frac{1 + K r}{r} - \frac{10 b}{R^2} \\
& \geq \frac{-2 \sqrt{10}}{R} \left( n + p - 2 \right) \frac{1 + K R}{R} - \frac{10 b}{R^2}, \\
& \geq -40 R^{-2} \left[ 2(n + p - 2)(1 + K R) + b \right].
\end{align*}
\]

here we use \( \theta' \leq 0 \), and \( \theta' = 0 \) if \( r \leq \frac{R}{2} \).
Thus (20) follows. ■

**Remark 14** The above lemma is the only place we need to assume that the sectional curvature of $M$ is bounded from below by $-K^2$.

**Lemma 15** The Hessian of $u$ satisfies that

\[
\sum_{i,j} u^2_{ij} \geq a_{n,p} f \eta^{-2} |\nabla \eta|^2 + \frac{1}{n-1} \left( \lambda_1^2(p-1)^{2(p-1)} f^{2-p} + \lambda^2(p-1)^{2(p-1)} + f^2 \right) + \frac{2(p-1)^{p-1}}{n-1} \left( \lambda f + \lambda_1 \lambda(p-1)^{(p-1)} f^{1-\frac{p}{2}} + \lambda_1 f^{2-\frac{p}{2}} \right) + \frac{p-1}{n-1} \cdot \langle \nabla \eta, \nabla u \rangle \eta^{-1} \left( \lambda_1(p-1)^{p-1} f^{1-\frac{p}{2}} + (p-1)^{p-1} \lambda + f \right)
\]

where $a_{n,p} = \frac{1+\min\{\frac{(p-1)^2}{n-1}, 1\}}{4}$.

**Proof.** Choose a local orthonormal frame \( \{e_i \} \) near any such given point $p_0$ so that, at $p_0$, \( \nabla u = |\nabla u| e_1 = u_1 e_1 \). Hence

\[
f_j = 2u_{j1}u_j = 2u_{j1}u_1 = 2u_{j1}\sqrt{f}
\]

for all $j$, and

\[
2u_{11} = \frac{f_j}{\sqrt{f}} = \frac{\langle \nabla f, \nabla u \rangle}{f}.
\]

Now we compute

\[
\sum_{i,j} u^2_{ij} \geq u^2_{11} + 2\sum_{k \geq 2} u^2_{kk} + \sum_{k \geq 2} u^2_{kk} \\
\geq u^2_{11} + 2\sum_{k \geq 2} u^2_{kk} + \frac{1}{n-1} (\sum_{k \geq 2} u_{kk})^2 \\
= u^2_{11} + 2\sum_{k \geq 2} u^2_{kk} + \frac{1}{n-1} (\Delta u - u_{11})^2.
\]
According to
\[
\Delta u = \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda f - \left( \frac{p}{2} - 1 \right) (\nabla u, \nabla f) f^{-1}
\]
\[
= \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f - \left( \frac{p}{2} - 1 \right) u_1 f_1 f^{-1}
\]
\[
= \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f - (p - 2) u_{11},
\]
then
\[
(\Delta u - u_{11})^2 = (\lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f - (p - 1) u_{11})^2
\]
\[
= (p - 1)^2 u_{11}^2 + \lambda_1^2 (p - 1)^2 (p - 1)^{2(p - 1)} f^{2 - p} + \lambda^2 (p - 1)^2 (p - 1)^{2(p - 1)} + f^2
\]
\[
- 2 (p - 1) u_{11} \left( \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f \right)
\]
\[
+ 2 (p - 1)^{p - 1} \lambda f + 2 \lambda_1 \lambda (p - 1)^2 (p - 1)^{2(p - 1)} f^{1 - \frac{p}{2}} + 2 \lambda_1 (p - 1)^{p - 1} f^{2 - \frac{p}{2}},
\]
and
\[
\left( 1 + \frac{(p - 1)^2}{n - 1} \right) u_{11}^2 + 2 \sum_{k \geq 2} u_{k1}^2 \geq 4 a_{n,p} \sum_{k \geq 1} u_{k1}^2,
\]
where \( a_{n,p} = \frac{1 + \min\{n - 1, \frac{(p - 1)^2}{n - 1} \}}{4} \). Hence (23) can be rewritten as
\[
\sum_{i,j} u_{ij}^2 \geq a_{n,p} \frac{\|\nabla f\|^2}{f} + \frac{1}{n - 1} \left( \lambda_1^2 (p - 1)^2 (p - 1)^{2(p - 1)} f^{2 - p} + \lambda^2 (p - 1)^2 (p - 1)^{2(p - 1)} + f^2 \right)
\]
\[
- \frac{p - 1}{n - 1} \frac{(\nabla f, \nabla u)}{f} \left( \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f \right)
\]
\[
+ \frac{2(p - 1)^{p - 1}}{n - 1} \left( \lambda f + \lambda_1 \lambda (p - 1)^2 (p - 1)^{2(p - 1)} f^{1 - \frac{p}{2}} + \lambda_1 f^{2 - \frac{p}{2}} \right)
\]
Thus, by using \( \nabla f = -f \eta^{-1} (\nabla \eta) \), the inequality becomes
\[
\sum_{i,j} u_{ij}^2 \geq a_{n,p} \frac{\|\nabla \eta\|^2}{\eta} + \frac{1}{n - 1} \left( \lambda_1^2 (p - 1)^2 (p - 1)^{2(p - 1)} f^{2 - p} + \lambda^2 (p - 1)^2 (p - 1)^{2(p - 1)} + f^2 \right)
\]
\[
+ \frac{2(p - 1)^{p - 1}}{n - 1} \left( \lambda f + \lambda_1 \lambda (p - 1)^2 (p - 1)^{2(p - 1)} f^{1 - \frac{p}{2}} + \lambda_1 f^{2 - \frac{p}{2}} \right)
\]
\[
+ \frac{p - 1}{n - 1} \frac{(\nabla \eta, \nabla u)}{\eta} \left( \lambda_1 (p - 1)^{p - 1} f^{1 - \frac{p}{2}} + (p - 1)^{p - 1} \lambda + f \right)
\]
Theorem 16 Suppose the sectional curvature of $(M, g)$ on a ball $B(x_0, R)$ satisfies $K_M \geq -K^2$. Let $v$ be a positive $C^2(B(x_0, R))$ solution of $\Delta_p v = -\lambda_1 |v|^{p-2} v - \lambda |\nabla v|^{p-2} v$ on a ball $B(x_0, R)$. Then for $R$ large enough,

$$\sup_{B(x_0, R/2)} |\nabla u|^2 \leq \frac{(n-1)^2 K^2 + C \left( R^{-1} + R^{-1/2} + R^{-2/(p+1)} \right)}{1 - CR^{-1}},$$

where $u = -(p-1) \log v$.

Proof. At the maximum point $p_0$ of $Q$, we are already proved that $\mathcal{L}(Q)$ is nonpositive, and we keep on its estimate

$$0 \geq \mathcal{L}(Q)$$

$$= \text{div} \left[ f^{\frac{p}{2}-1} \cdot A(\nabla (\eta f)) \right]$$

$$= \text{div} \left[ f^{\frac{p}{2}-1} \cdot \left( A(\nabla \eta) \cdot f + A(\nabla f) \cdot \eta \right) \right]$$

$$= \text{div} \left[ f^{\frac{p}{2}} A(\nabla \eta) \right] + \text{div} \left[ f^{\frac{p}{2}-1} A(\nabla f) \cdot \eta \right]$$

$$= \eta \mathcal{L}(f) + f^{\frac{p}{2}-1} \langle A(\nabla f), \nabla \eta \rangle + \frac{p}{2} f^{\frac{p}{2}-1} \langle A(\nabla \eta), \nabla f \rangle + f^{\frac{p}{2}} \text{div} A(\nabla \eta)$$

$$\geq \eta \mathcal{L}(f) - \frac{p}{2} f^{\frac{p}{2}-1} \langle A(\nabla \eta), \nabla \eta \rangle + f^{\frac{p}{2}} \text{div} A(\nabla \eta)$$

here we use $\nabla f = -f \eta^{p-1}(\nabla \eta)$. Product the term $f^{\frac{p}{2}-1} \eta^{p-1}$ to both sides of the inequality, then one has

$$0 \geq f^{\frac{p}{2}-1} \eta^{p-1} \mathcal{L}(f) - \left( \frac{p}{2} + 1 \right) f^{p-1} \eta^{p-2} \langle A(\nabla \eta), \nabla \eta \rangle + f^{p-1} \eta^{p-1} \text{div} A(\nabla \eta).$$

If $1 < p \leq 2$ we have $\langle A(\nabla \eta), \nabla \eta \rangle \leq |\nabla \eta|^2$, and the case $p \geq 2$ we have $\langle A(\nabla \eta), \nabla \eta \rangle \leq (p-1)|\nabla \eta|^2$, then one obtains $\langle A(\nabla \eta), \nabla \eta \rangle \leq b|\nabla \eta|^2$ where $b = \max\{1, p-1\}$. So (24) becomes

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\[ \frac{3}{2} - 1 \eta^p \mathcal{L}(f) - \left( \frac{p}{2} + 1 \right) b f^{p-1} \eta^{p-2} |\nabla \eta|^2 + f^{p-1} \eta^{p-1} \operatorname{div} A(\nabla \eta). \]

Now, combining lemma 12, lemma 13 and lemma 15. We may rewrite (25) as, by \((\nabla u, \nabla \eta) \geq -|\nabla \eta|f^{\frac{1}{2}}\) and \(R_{ij} u_i u_j \geq -(n - 1)K^2 f\),

\[
0 \geq f^{\frac{p}{2} - 1} \eta^p \mathcal{L}(f) - \left( \frac{p}{2} + 1 \right) b f^{p-1} \eta^{p-2} |\nabla \eta|^2 + f^{p-1} \eta^{p-1} \operatorname{div} A(\nabla \eta) \\
\geq (\frac{p}{2} - 1) f^{p-1} \eta^{p-2} |\nabla \eta|^2 + 2 \alpha_{n,p} f^{p-1} \eta^{p-2} |\nabla \eta|^2 - 2(n - 1)K^2 f^{p-1} \eta^p \\
+ \frac{2}{n - 1} \left( \lambda^2 (p - 1)^{2(p-1)} \eta^p + \lambda^2 (p - 1)^{2(p-1)} f^{p-2} \eta^p + f^p \eta^p \right) \\
+ \frac{4(p-1)^{p-1}}{n - 1} \left( \lambda f^{p-1} \eta^p + \lambda_1 \lambda (p - 1)^{(p-1)} f^{\frac{p}{2} - 1} \eta^p + \lambda_1 f^{\frac{p}{2}} \eta^p \right) \\
- \frac{2(p - 1)}{n - 1} [\nabla \eta] \left( \lambda_1 (p - 1)^{p-1} f^{\frac{p}{2} - 1/2} \eta^{p-1} + (p - 1)^{p-1} \lambda f^{p-3/2} \eta^{p-1} + f^{p-1/2} \eta^{p-1} \right) \\
- p f^{p-1/2} \eta^{p-1} |\nabla \eta| - \lambda (p - 1)^{p-1} |p - 2| f^{p-3/2} \eta^{p-1} |\nabla \eta| - (\frac{p}{2} + 1) b f^{p-1} \eta^{p-2} |\nabla \eta|^2 \\
- 40R^{-2} [2(n + p - 2)(1 + KR) + b] f^{p-1} \eta^{p-1} - \frac{p}{2} |p - 2| f^{p-1} \eta^{p-2} |\nabla \eta|^2 \\
- (\frac{p}{2} - 1) f^{p-1} \eta^{p-2} |\nabla \eta|^2 - \lambda_1 |p - 2| (p - 1)^{p-1} f^{\frac{p}{2} - 1/2} \eta^{p-1} |\nabla \eta| \\
- \lambda (p - 1)^{p-1} |p - 2| f^{p-3/2} \eta^{p-1} |\nabla \eta| - |p - 2| f^{p-1/2} \eta^{p-1} |\nabla \eta| \\
\text{for which we have} \]

\[
0 \geq \frac{2}{n - 1} \left( \lambda^2 (p - 1)^{2(p-1)} \eta^p + \lambda^2 (p - 1)^{2(p-1)} f^{p-2} \eta^p + f^p \eta^p \right) \\
+ \frac{4(p-1)^{p-1}}{n - 1} \left( \lambda f^{p-1} \eta^p + \lambda_1 \lambda (p - 1)^{(p-1)} f^{\frac{p}{2} - 1} \eta^p + \lambda_1 f^{\frac{p}{2}} \eta^p \right) \\
- 2(n - 1)K^2 f^{p-1} \eta^p - \frac{C}{R} \left( (f \eta)^{p-1} + (f \eta)^{\frac{p}{2} - 1/2} + (f \eta)^{p-3/2} + (f \eta)^{p-1/2} \right) \\
\text{Then, using } Q = \eta f, \text{ it becomes} \]

\[
0 \geq \frac{2}{n - 1} \left( \lambda^2 (p - 1)^{2(p-1)} \eta^p + \lambda^2 (p - 1)^{2(p-1)} Q^{p-2} \eta^2 + Q^p \right) \\
+ \frac{4(p-1)^{p-1}}{n - 1} \left( \lambda Q^{p-1} \eta + \lambda_1 \lambda (p - 1)^{(p-1)} Q^{\frac{p}{2} - 1} \eta^{p/2+1} + \lambda_1 Q^{\frac{p}{2}} \eta^{p/2} \right) \\
- 2(n - 1)K^2 Q^{p-1} - \frac{C}{R} \left( Q^{p-1} + Q^{\frac{p}{2} - 1/2} + Q^{p-3/2} + Q^{p-1/2} \right) \\
\]

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which implies

\[
0 \geq \frac{2}{n-1}Q^p - 2(n-1)K^2Q^{p-1} \\
- \frac{C}{R} \left( Q^{p-1} + Q^{\frac{p}{2}-1/2} + Q^{p-3/2} + Q^{p-1/2} \right).
\]

So we obtain

\[
0 \geq \left( \frac{2}{n-1} - \frac{C}{R} \right) Q^p - \left( 2(n-1)K^2 + \frac{C}{R} \right) Q^{p-1} \\
- \frac{C}{R} \left( Q^{\frac{p}{2}-1/2} + Q^{p-3/2} \right)
\]

i.e.

\[
0 \geq \left( \frac{2}{n-1} - \frac{C}{R} \right) Q^p - \left( 2(n-1)K^2 + \frac{C}{R} \right) Q^{p-1} - \frac{C}{R}Q^{p-2} - \frac{C}{R}Q^{\frac{p-1}{2}}.
\]

Now we let \( K = 0 \).

If \( Q \geq 1 \), one has

\[
0 \geq \left( \frac{2}{n-1} - \frac{C}{R} \right) Q^p - \left( 2(n-1)K^2 + \frac{C}{R} \right) Q^{p-1}
\]

i.e.

\[
0 \geq \left( \frac{2}{n-1} - \frac{C}{R} \right) Q - \left( 2(n-1)K^2 + \frac{C}{R} \right).
\]

For which we have

\[
Q(p) \leq \frac{(n-1)^2K^2 + \frac{C}{R}}{1 - \frac{C}{R}}.
\]

If \( Q < 1 \) and \( p - 2 < \frac{p-1}{2} \), then (27) can be rewritten as

\[
\left( \frac{2}{n-1} - \frac{C}{R} \right) Q^p \leq \frac{C}{R} \left( Q^{p-1} + Q^{p-2} + Q^{\frac{p-1}{2}} \right) \leq \frac{C}{R}Q^{p-2},
\]

then we obtain

\[
Q \leq \frac{C}{\sqrt{R}}.
\]
If \( Q < 1 \) and \( p - 2 \geq \frac{p - 1}{2} \), then (27) can be rewritten as

\[
\left(\frac{2}{n-1} - \frac{C}{R}\right) Q^p \leq \frac{C}{R} \left( Q^{p-1} + Q^{p-2} + Q^{\frac{p-1}{2}} \right) \leq \frac{C}{R} Q^{\frac{p-1}{2}} ,
\]

then

\[
Q \leq \frac{C}{R^{\frac{2}{2-p}}} .
\]

Now we assume \( K > 0 \).

If \( Q > A = \frac{(n-1)^2 K^2 + C}{1 - \frac{C}{R}} \), then (27) implies

\[
Q^p \leq AQ^{p-1} + \frac{C}{R} Q^{p-2} + \frac{C}{R} Q^{\frac{p-1}{2}} ,
\]

so, for \( R \) large enough,

\[
Q \leq A + \frac{C}{R} Q^{-1} + \frac{C}{R} Q^{-\frac{p-1}{2}} \leq A + \frac{C}{R} A^{-1} + \frac{C}{R} A^{-\frac{p-1}{2}} \leq A.
\]

Hence, by (28), (29), (30) and (31), we conclude that

\[
\sup_{B(x_0, R/2)} |\nabla u|^2 \leq \frac{(n-1)^2 K^2 + C}{1 - \frac{C}{R}} ,
\]

and this completes the proof.

\[\blacksquare\]

**Corollary 17** Suppose the sectional curvature of \((M, g)\) on a ball \(B(x_0, R)\) satisfies \( K_M \geq -K^2 \) with \( K > 0 \). Let \( v \) be a positive \( C^3 \) \((B(x_0, R))\) solution of \( \Delta_g v = -\lambda_1 |v|^{p-2} v - \lambda |\nabla v|^{p-2} v \).

Then for \( R \) large enough,

\[
\sup_{B(x_0, R/2)} |\nabla u|^2 \leq \frac{(n-1)^2 K^2 + C}{1 - \frac{C}{R}} ,
\]

where \( u = -(p - 1) \log v \).
References

[1] S-C Chang, J-T Chen, S. W. Wei, Liouville properties for p-harmonic maps with finite q-energy, arXiv:1211.2899 accepted by Transactions of the AMS


