The case for \( k \geq 4 \)

By using the Method of Infinite Decent, we can prove Theorem 2.1.

**Theorem 2.1** The Diophantine equation \( x^2 + y^2 + z^2 = kxyz \) has no positive integer solution, where \( k \geq 4 \), \( k \in \mathbb{N} \).

**Proof.** First, we claim that the entries of the solution \((a, b, c)\) are all distinct. We suppose that \( a = b \), then \( 2a^2 + c^2 = ka^2c \). Hence \( a \mid c \). Let \( c = na, \ n \in \mathbb{N} \), we can get \( 2 + n^2 = nka \). This implies \( n \mid 2 \), thus \( n = 1 \) or \( 2 \), contradicting to the equation.

Next, let \((a, b, c)\) be a solution. It is easy to see that \( c \) and \( c' = kab - c \) are actually the roots of the quadratic equation \( z^2 - kabz + a^2 + b^2 = 0 \) in \( z \). It follows that \( cc' = a^2 + b^2 \). In particular, \( c' > 0 \). Thus, \((a, b, kab - c)\) is indeed a solution.

We consider that

\[
(e - b)(e' - b) = cc' - (c + c')b + b^2 = a^2 + b^2 - kab^2 + b^2 = a^2 + 2b^2 - kab^2 < 3b^2 - kab^2 = b^2(3 - ka) < 0
\]

By the Method of Infinite Descent, we complete the proof of Theorem 2.1. \( \square \)