# 3 The Second Form : $\quad Y^b + Y^c Z^d$

Again, let $K$ be a field of characteristic $p > 0$ and

$$S = K\left[Y_1, \ldots, Y_s, Z_1, \ldots, Z_t\right].$$

In this section we shall determine the Hilbert-Kunz function of the hypersurface of the following form :

$$f := Y^b + Y^c Z^d$$

where $Y^b = Y_1^{b_1} \ldots Y_s^{b_s}$, $Y^c = Y_1^{c_1} \ldots Y_s^{c_s}$, $Z^d = Z_1^{d_1} \ldots Z_t^{d_t}$. Let $q = p^n$, $J = \{j \mid b_j > c_j\}$, and set $R = S/ < f >$. Then $0 \leq |J| := m \leq s$, and *w.l.o.g.*, we assume that $b_1 > c_1, \ldots,\ b_m > c_m,\ b_{m+1} \leq c_{m+1}, \ldots,\ b_s \leq c_s$. We shall determine the assignment

$$HK_R(q) := dim_K\Big(\ S\ /< Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, f >\Big).$$

Throughout this section, it is not restrictive to assume that $b_1 - c_1 \geq b_2 - c_2 \geq \cdots \geq b_m - c_m > 0$ , $c_{m+1} - b_{m+1} \geq c_{m+2} - b_{m+2} \geq \cdots \geq c_s - b_s \geq 0$, and $d_1 \geq d_2 \geq \cdots \geq d_t > 0$. Let $u$ be the maximum of the integers $b_1 - c_1$, $c_{m+1} - b_{m+1}$, and $d_1$; that is, $u$ is the greatest integer among all $(b_j - c_j)$'s , $(c_h - b_h)$'s, and $d_k$'s. We also denote by $[y]$ the greatest integer less than or equal to $y$, and $S_{in}(x)$ the elementary symmetric polynomial of degree $i$ in $n$ indeterminates $x = (x_1, \ldots, x_n)$. Let $I_q$ be the ideal of $S$ generated by all $Y_j^q$'s, $Z_k^q$'s, and $f$, and define $(v)_+ = max\{0, v\}$.

Firstly, we prove the following lemma.

**Lemma 3.1.** *Let* $S = K[Y_1, \dots, Y_s]$, $s \geq 1$, *and* $G$ *the ideal generated by*

$$Y_1^q, \dots, Y_s^q, Y_1^{[q-\alpha(b_1-c_1)-c_1]_+} Y^{e+\alpha(c-b)_+}, \dots, Y_m^{[q-\alpha(b_m-c_m)-c_m]_+} Y^{e+\alpha(c-b)_+}, \quad \text{and } Y^b,$$

*where* $\alpha$ *is a positive integer,* $e = (e_1, \dots, e_s)$, $e_1 = c_1, \dots, e_m = c_m, e_{m+1} = b_{m+1}, \dots, e_s = b_s$, *and* $(c-b)_+ = (0, \dots, 0, c_{m+1} - b_{m+1}, \dots, c_s - b_s)$. *Then the dimension of* $S/G$ *is equal to*

$$\begin{aligned} q^s - \prod_{j=1}^{s}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\ + \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\ + \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\ - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)-c_j]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ . \end{aligned}$$

**Proof :** If $[q-\alpha(b_j-c_j)-c_j\,]_+ = 0$ for some $j$ with $1 \leq j \leq m$, then $G$ is generated by

$$Y_1^q, \dots, Y_s^q, \; Y^{e+\alpha(c-b)_+}, \quad \text{and} \quad Y^b.$$

Thus,

$$\begin{aligned} dim_K\big(S/G\big) = q^s - \prod_{j=1}^{s}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\ + \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ . \end{aligned}$$

From now on, we assume $q-\alpha(b_j-c_j)-c_j > 0$ for each $j = 1, 2, \dots, m$. Let $l_\alpha$ be the minimum of

$$\left\{ \left[\frac{q-\alpha(b_j-c_j)-1}{c_j}\right], \left[\frac{q-1}{b_h+\alpha(c_h-b_h)}\right] \;\middle|\; j = 1, \dots, m, \; h = m+1, \dots, s \right\}.$$

Then we have $\quad q-\alpha(b_{j_0}-c_{j_0}) \leq (l_\alpha+1)c_{j_0}\quad$ for some $j_0$ with $1 \leq j_0 \leq m \quad$ or

$q \leq (l_\alpha+1)\big(b_{h_0}+\alpha(c_{h_0}-b_{h_0})\big)\quad$ for some $h_0$ with $m+1 \leq h_0 \leq s$, $\quad$ and

$q-\alpha(b_j-c_j)-l_\alpha c_j \geq 1\quad$ for each $j\quad$ and $\quad q-l_\alpha(b_h+\alpha(c_h-b_h)) \geq 1\quad$ for each $h$.

We consider the ideals $G_\beta = G : Y^{\beta[e+\alpha(c-b)_+]}$, for $\beta = 0, 1, 2, \ldots, l_\alpha + 1$. Since $G_0 = G$, $G_{l_\alpha+1} = S$, and $G_{\beta+1} = G_\beta : Y^{e+\alpha(c-b)_+}$, we have the exact sequence of K-modules :

$$0 \longrightarrow S/G_{\beta+1} \overset{Y^{e+\alpha(c-b)_+}}{\longrightarrow} S/G_\beta \longrightarrow S \ / < G_\beta, Y^{e+\alpha(c-b)_+} > \longrightarrow 0.$$

It follows that

$$dim_K\big(S/G\big) = dim_K\big(S/G_0\big) = \sum_{\beta=0}^{l_\alpha} dim_K\Big(\ S \ / < G_\beta, Y^{e+\alpha(c-b)_+} > \Big).$$

We compute $dim_K\Big(S \ / < G_\beta, Y^{e+\alpha(c-b)_+} >\Big)$ as follows :
For $\beta = 0$, the ideal $< G_0, Y^{e+\alpha(c-b)_+} >$ is generated by $Y_1^q, \ldots, Y_s^q$, $Y^{e+\alpha(c-b)_+}$, and $Y^b$.
Thus,

$$\begin{aligned} dim_K\Big(S \ / < G_0, Y^{e+\alpha(c-b)_+} >\Big) = q^s - \prod_{j=1}^{s}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\ + \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \,. \end{aligned}$$

For $1 \leq \beta \leq l_\alpha$, the ideal $G_\beta = G : Y^{\beta[e+\alpha(c-b)_+]}$ is generated by

$Y_1^{q-\alpha(b_1-c_1)-\beta c_1}, \ldots, Y_m^{q-\alpha(b_m-c_m)-\beta c_m}, Y_{m+1}^{q-\beta[b_{m+1}+\alpha(c_{m+1}-b_{m+1})]}, \ldots, Y_s^{q-\beta[b_s+\alpha(c_s-b_s)]}$, and

$Y_1^{(b_1-\beta c_1)_+} \cdots Y_m^{(b_m-\beta c_m)_+} Y_{m+1}^{[b_{m+1}-\beta(b_{m+1}+\alpha(c_{m+1}-b_{m+1}))]_+} \cdots Y_s^{[b_s-\beta(b_s+\alpha(c_s-b_s))]_+}$ .

Thus, the ideal $< G_\beta, Y^{e+\alpha(c-b)_+} >$ is generated by

$Y_1^{q-\alpha(b_1-c_1)-\beta c_1}, \ldots, Y_m^{q-\alpha(b_m-c_m)-\beta c_m}, Y_{m+1}^{q-\beta[b_{m+1}+\alpha(c_{m+1}-b_{m+1})]}, \ldots, Y_s^{q-\beta[b_s+\alpha(c_s-b_s)]}$,

$Y_1^{(b_1-\beta c_1)_+} \cdots Y_m^{(b_m-\beta c_m)_+} Y_{m+1}^{[b_{m+1}-\beta(b_{m+1}+\alpha(c_{m+1}-b_{m+1}))]_+} \cdots Y_s^{[b_s-\beta(b_s+\alpha(c_s-b_s))]_+}$, and $Y^{e+\alpha(c-b)_+}$.

Hence,

$$\begin{aligned} & dim_K\Big(S \ / < G_\beta, Y^{e+\alpha(c-b)_+} >\Big) \\ & = \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j]_+ \prod_{h=m+1}^{s} [q-\beta(b_h+\alpha(c_h-b_h))]_+ \\ & \quad - \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-(\beta+1)c_j]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \\ & \quad - \prod_{j=1}^{m} \big[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+\big]_+ \prod_{h=m+1}^{s} \big[q-\beta(b_h+\alpha(c_h-b_h))-[b_h-\beta(b_h \\ & \quad +\alpha(c_h-b_h))]_+\big]_+ + \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \,, \end{aligned}$$

where $u_{j\beta} = max\{\ c_j\ ,\ [\ b_j - \beta c_j\ ]_+\ \}$.

Now,we have

$$
\begin{aligned}
dim_K\big(S/G\big) = q^s - \prod_{j=1}^{s}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
+ \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
+ \sum_{\beta=1}^{l_\alpha} \Bigg\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j\ ]_+ \prod_{h=m+1}^{s} [q-\beta(b_h+\alpha(c_h-b_h))]_+ \\
- \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-(\beta+1)c_j\ ]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \Bigg\} \\
- \sum_{\beta=1}^{l_\alpha} \Bigg\{ \prod_{j=1}^{m} \big[q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+\big]_+ \prod_{h=m+1}^{s} [q-\beta(b_h+\alpha(c_h-b_h))]_+ \\
- \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \Bigg\}.
\end{aligned}
$$

Let $(\triangle)$ be the term

$$
\begin{aligned}
\sum_{\beta=1}^{l_\alpha} \Bigg\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j\ ]_+ \prod_{h=m+1}^{s} [q-\beta(b_h+\alpha(c_h-b_h))]_+ \\
- \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-(\beta+1)c_j\ ]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \Bigg\}.
\end{aligned}
$$

Since $q-\alpha(b_{j_0}-c_{j_0}) \leq (l_\alpha+1)c_{j_0}$ for some $j_0$ with $1 \leq j_0 \leq m$ or
$q \leq (l_\alpha+1)\big(b_{h_0}+\alpha(c_{h_0}-b_{h_0})\big)$ for some $h_0$ with $m+1 \leq h_0 \leq s$,

the term $(\triangle)$ is equal to $\displaystyle\prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j\ ]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+$ .

Let $(\triangle\triangle)$ be the term

$$
\begin{aligned}
\sum_{\beta=1}^{l_\alpha} \Bigg\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j-(b_j-\beta c_j)_+]_+ \prod_{h=m+1}^{s} [q-\beta(b_h+\alpha(c_h-b_h))]_+ \\
- \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}\ ]_+ \prod_{h=m+1}^{s} [q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+ \Bigg\}.
\end{aligned}
$$

Since

$$\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-\beta c_j-u_{j\beta}\ ]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+$$
$$=\prod_{j=1}^{m}\left[q-\alpha(b_j-c_j)-(\beta+1)c_j-(b_j-(\beta+1)c_j)_+\right]_+\prod_{h=m+1}^{s}[q-(\beta+1)(b_h+\alpha(c_h-b_h))]_+\,,$$

where $\beta=1,2,\ldots,l_\alpha-1$, the term $(\triangle\triangle)$ is equal to

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+$$
$$-\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-l_\alpha c_j-u_{jl_\alpha}\ ]_+\prod_{h=m+1}^{s}[q-(l_\alpha+1)(b_h+\alpha(c_h-b_h))]_+\,.$$

Since $q-\alpha(b_{j_0}-c_{j_0})\leq(l_\alpha+1)c_{j_0}$ or $q\leq(l_\alpha+1)\big(b_{h_0}+\alpha(c_{h_0}-b_{h_0})\big)$, we have

$$\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-l_\alpha c_j-u_{jl_\alpha}\ ]_+=0\quad\text{or}\quad\prod_{h=m+1}^{s}[q-(l_\alpha+1)(b_h+\alpha(c_h-b_h))]_+=0.$$

Thus, $(\triangle\triangle)$ is equal to

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\,.$$

Therefore,

$$\begin{aligned}dim_K\big(S/G\big)=q^s-\prod_{j=1}^{s}(q-b_j)-\prod_{j=1}^{m}(q-c_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
+\prod_{j=1}^{m}(q-b_j)\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
+\prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\\
-\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\,.\end{aligned}$$

□

Since $u$ is the maximum of the integers among all $(b_j - c_j)$'s, $(c_h - b_h)$'s, and $d_k$'s, we have

$$\left[\frac{q-v}{u}\right] := min\left\{\left[\frac{q-c_j-1}{b_j-c_j}\right], \left[\frac{q-b_h-1}{c_h-b_h}\right], \left[\frac{q-1}{d_k}\right] \;\middle|\; \begin{array}{l} b_j - c_j > 0, \quad c_h - b_h > 0, \;\; 1 \le j \le m \\ m+1 \le h \le s, \;\; 1 \le k \le t \end{array}\right\}$$

for $q \gg 0$, where $v = 1$ or $1 + c_j$ for some $j$ or $1 + b_h$ for some $h$. Let $l_u$ be the integer $\left[\frac{q-v}{u}\right]$, and $\epsilon$ be the remainder of $q - v$ divided by $u$. Then $l_u = \frac{q-v-\epsilon}{u}$ and one has $q - l_u(b_j - c_j) - c_j > 0$, $q - l_u(c_h - b_h) - b_h > 0$, and $q - l_u d_k > 0$ for all $j$, $h$, and $k$. On the other hand, by the definition of $l_u$, at least one of $[q - (l_u + 1)(b_j - c_j) - c_j\,]_+$'s , $[q - (l_u + 1)(c_h - b_h) - b_h]_+$'s , and $[q - (l_u + 1)d_k]_+$'s must be zero .

**Proposition 3.2.** *Let* $f := Y^b + Y^c Z^d$. *Then* $HK_R(q)$ *is equal to*

$$q^{s+t} - q^t \prod_{j=1}^{s}(q - b_j)$$
$$-\prod_{j=1}^{m}(q - c_j) \times \left\{\sum_{\alpha=1}^{l_u}\left[\prod_{h=m+1}^{s}[q - \alpha(c_h - b_h) - b_h]\right] \times \left[\prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t}[q - (\alpha+1)d_k]_+\right]\right\}$$
$$+\prod_{j=1}^{m}(q - b_j) \times \left\{\sum_{\alpha=1}^{l_u}\left[\prod_{h=m+1}^{s}[q - \alpha(c_h - b_h) - b_h]\right] \times \left[\prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t}[q - (\alpha+1)d_k]_+\right]\right\}$$
$$+\sum_{\alpha=1}^{l_u}\left\{\prod_{j=1}^{m}[q - \alpha(b_j - c_j) - c_j\,] \prod_{h=m+1}^{s}[q - \alpha(c_h - b_h) - b_h] - \prod_{j=1}^{m}[q - (\alpha+1)(b_j - c_j) - c_j\,]_+\right.$$
$$\left.\prod_{h=m+1}^{s}[q - \alpha(c_h - b_h) - b_h]\right\} \times \left\{\prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t}[q - (\alpha+1)d_k]_+\right\},$$

*where* $l_u$ *is the integer* $\left[\frac{q-v}{u}\right]$, *and* $0 \le m \le s$ .

**Proof :** We prove this proposition by discussing on $m$, and let $<$ be the lexicographic order on $S$.

*Case 1 :* Assume that $m = 0$, *i.e.*, $b_j \le c_j$ for each $j = 1, 2, \ldots, s$. Then $Y^c Z^d$ is the leading term of $f$. The elements

$$Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, \text{ and } Y^b$$

form a Gröbner basis of the ideal $I_q$ . Thus, the ideal $in(I_q)$ is generated by the elements as above. It follows that

$$dim_K\big(S/in\,(I_q)\big) = dim_K\Big(S\;/ < Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, Y^b >\Big).$$

Hence, $HK_R(q) = q^{s+t} - q^t \prod_{j=1}^{s}(q-b_j)$ .

*Case 2 :* Suppose that $1 \leq m \leq s$, and define

$$e_j = c_j \ \text{ for } \ 1 \leq j \leq m, \quad \text{and} \quad e_h = b_h \ \text{ for } \ m+1 \leq h \leq s.$$

Then $Y^b$ is the leading term of $f$ and $Y^e = Y_1^{e_1} \cdots Y_s^{e_s} = Y_1^{c_1} \cdots Y_m^{c_m} Y_{m+1}^{b_{m+1}} \cdots Y_s^{b_s}$ .
By means of Buchberger's algorithm (Algorithm 1.9), the elements

$$Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, \ Y^b + Y^c Z^d, \ \text{ and}$$
$$Y_j^{[q-\delta(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+} Z^{\delta d}, \ j = 1, \ldots, m, \ \delta = 1, \ldots, l,$$

form a Gröbner basis of the ideal $I_q$, where $l = \left[\frac{q-1}{d_1}\right]$.
Hence, the ideal $in(I_q)$ is generated by

$$Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, \ Y^b, \text{ and } \ Y_j^{[q-\delta(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+} Z^{\delta d}, \ j = 1, \ldots, m, \ \delta = 1, \ldots, l.$$

Now we have to compute the dimension of $S/in(I_q)$. In order to do this, we consider the ideals $K_\alpha = in(I_q) : Z^{\alpha d}$ for $\alpha = 0, 1, \ldots, l+1$. Since $K_0 = in(I_q)$, $K_{l+1} = S$, and $K_{\alpha+1} = K_\alpha : Z^d$, we have the exact sequence of K-modules :

$$0 \longrightarrow S/K_{\alpha+1} \stackrel{Z^d}{\longrightarrow} S/K_\alpha \longrightarrow S \ / < K_\alpha, Z^d > \longrightarrow 0.$$

It follows that

$$dim_K\big(S/in(I_q)\big) = dim_K\big(S/K_0\big) = \sum_{\alpha=0}^{l} dim_K\Big(S \ / < K_\alpha, Z^d >\Big).$$

We compute $dim_K\Big(S \ / < K_\alpha, Z^d >\Big)$ as follows :

For $\alpha = 0$, the ideal $< K_0, Z^d >$ is generated by $Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, \ Y^b$, and $Z^d$.
Therefore,

$$dim_K\Big(S \ / < K_0, Z^d >\Big) = dim_K\Big(S \ / < Y_1^q, \ldots, Y_s^q, Z_1^q, \ldots, Z_t^q, Y^b, Z^d >\Big)$$
$$= q^{s+t} - q^t \prod_{j=1}^{s}(q-b_j) - q^s \prod_{k=1}^{t}(q-d_k) + \prod_{j=1}^{s}(q-b_j) \prod_{k=1}^{t}(q-d_k).$$

For $1 \leq \alpha \leq l$, the ideal $< K_\alpha, Z^d >$ is generated by

$$Y_1^q, \ldots, Y_s^q, Z_1^{q-\alpha d_1}, \ldots, Z_t^{q-\alpha d_t}, \ Y^b, \ Z^d, \text{ and } \ Y_j^{[q-\alpha(b_j-c_j)-c_j]_+} Y^{e+\delta(c-b)_+}, \ j = 1, \ldots, m.$$

Let $S_1 = K[Y_1, \ldots, Y_s]$, and $S_2 = K[Z_1, \ldots, Z_t]$. Then by Lemma 3.1, we have

$$
\begin{aligned}
& dim_K\Big(S \;/ < K_\alpha, Z^d >\Big) \\
& = dim_K\Big(S_1 \;/ < Y_1^q, \ldots, Y_s^q, Y^b, Y_1^{[q-\alpha(b_1-c_1)-c_1]_+} Y^{e+\alpha(c-b)_+}, \ldots, Y_m^{[q-\alpha(b_m-c_m)-c_m]_+} Y^{e+\alpha(c-b)_+} >\Big) \\
& \quad \times dim_K\Big(S_2 \;/ < Z_1^{q-\alpha d_1}, \ldots, Z_t^{q-\alpha d_t}, Z^d >\Big) \\
& = \Bigg\{ \; q^s - \prod_{j=1}^{s}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
& \qquad\qquad + \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
& \qquad\qquad + \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j\,]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \\
& \qquad\qquad - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)-c_j\,]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \\
& \quad \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k]_+ \; \Bigg\}.
\end{aligned}
$$

Since $dim_K\big(S/in(I_q)\big)$ can be written as

$$
dim_K\Big(S \;/ < K_0, Z^d >\Big) \; + \; \sum_{\alpha=1}^{l} dim_K\Big(S \;/ < K_\alpha, Z^d >\Big),
$$

it follows that $dim_K\big(S/in(I_q)\big)$ is equal to

$$
\begin{aligned}
& q^{s+t} - q^t \prod_{j=1}^{s}(q-b_j) - q^s \prod_{k=1}^{t}(q-d_k) + \prod_{j=1}^{s}(q-b_j) \prod_{k=1}^{t}(q-d_k) \\
& + \Bigg[ \; q^s - \prod_{j=1}^{s}(q-b_j) \Bigg] \times \Bigg\{ \sum_{\alpha=1}^{l} \Bigg[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k]_+ \; \Bigg] \Bigg\} \\
& - \sum_{\alpha=1}^{l} \Bigg\{ \prod_{j=1}^{m}(q-c_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k]_+ \Bigg\} \\
& + \sum_{\alpha=1}^{l} \Bigg\{ \prod_{j=1}^{m}(q-b_j) \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k]_+ \Bigg\} \\
& + \sum_{\alpha=1}^{l} \Bigg\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j]_+ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)-c_j]_+ \\
& \qquad \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]_+ \Bigg\} \times \Bigg\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k]_+ \Bigg\}.
\end{aligned}
$$

Let $(\triangle\triangle\triangle)$ be the term

$$\left[\ q^s - \prod_{j=1}^{s}(q-b_j)\ \right] \times \left\{\ \sum_{\alpha=1}^{l} \left[\ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\ \right]\ \right\}.$$

Since $l = \left[\frac{q-1}{d_1}\right]$, we have $q \le (l+1)d_1$, and so $\prod_{k=1}^{t}[q-(\alpha+1)d_k]_+ = 0$.
Thus, the term $(\triangle\triangle\triangle)$ is equal to

$$q^s\prod_{k=1}^{t}(q-d_k) - \prod_{j=1}^{s}(q-b_j)\prod_{k=1}^{t}(q-d_k).$$

It follows that $dim_K\big(S/in(I_q)\big)$ is equal to

$$\begin{aligned} & q^{s+t} - q^t\prod_{j=1}^{s}(q-b_j) \\ & -\prod_{j=1}^{m}(q-c_j) \times \left\{\sum_{\alpha=1}^{l}\left[\ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\right] \times \left[\prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\right]\right\} \\ & +\prod_{j=1}^{m}(q-b_j) \times \left\{\sum_{\alpha=1}^{l}\left[\ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\right] \times \left[\prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\right]\right\} \\ & +\sum_{\alpha=1}^{l}\left\{\ \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j]_+\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+ - \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j]_+\right. \\ & \left.\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]_+\right\} \times \left\{\ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\ \right\}. \end{aligned}$$

By the definition of $l_u$, we have

$$\begin{aligned} & HK_R(q) = q^{s+t} - q^t\prod_{j=1}^{s}(q-b_j) \\ & -\prod_{j=1}^{m}(q-c_j) \times \left\{\sum_{\alpha=1}^{l_u}\left[\ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]\right] \times \left[\prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\right]\right\} \\ & +\prod_{j=1}^{m}(q-b_j) \times \left\{\sum_{\alpha=1}^{l_u}\left[\ \prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]\right] \times \left[\prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\right]\right\} \\ & +\sum_{\alpha=1}^{l_u}\left\{\ \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ]\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h] - \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]_+\right. \\ & \left.\prod_{h=m+1}^{s}[q-\alpha(c_h-b_h)-b_h]\right\} \times \left\{\ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k]_+\ \right\}. \end{aligned}$$

Finally, combining *the Case 1* with *the Case 2* , and we have the following result :

$$HK_R(q) = q^{s+t} - q^t \prod_{j=1}^{s}(q - b_j)$$

$$- \prod_{j=1}^{m}(q - c_j) \times \left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right] \times \left[ \prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k]_+ \right] \right\}$$

$$+ \prod_{j=1}^{m}(q - b_j) \times \left\{ \sum_{\alpha=1}^{l_u} \left[ \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right] \times \left[ \prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k]_+ \right] \right\}$$

$$+ \sum_{\alpha=1}^{l_u} \left\{ \prod_{j=1}^{m} [q - \alpha(b_j - c_j) - c_j\ ] \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] - \prod_{j=1}^{m} [q - (\alpha+1)(b_j - c_j) - c_j\ ]_+ \right.$$

$$\left. \prod_{h=m+1}^{s} [q - \alpha(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q - \alpha d_k) - \prod_{k=1}^{t} [q - (\alpha+1)d_k]_+ \right\} .$$

where $0 \le m \le s$ . □

We will need some lemmas before observing the behavior of the Hilbert-Kunz function of $R$.

**Lemma 3.3.**

$$q^{s+t} - q^t \prod_{j=1}^{s}(q - b_j) = S_{1s}(b)q^{s+t-1} \ + \ (\ \textit{terms of degree} \le s+t-2 \ \textit{ in } \ q \ \textit{ over } \ Z\ ),$$

*where* $b = (b_1, b_2, \ldots, b_s)$.

**Proof :** The proof is similar to Lemma 2.3 . □

**Lemma 3.4.**

$$\prod_{j=1}^{m}(q - b_j) - \prod_{j=1}^{m}(q - c_j) = -S_{1m}(b' - c')q^{m-1} + (\ \textit{terms of degree} \ \le \ m-2 \ \textit{ in } \ q \ \textit{ over } \ Z\ ),$$

*where* $b' - c' = (b_1 - c_1, \ldots, b_m - c_m)$.

**Proof :** The proof is similar to Lemma 2.4 . □

**Lemma 3.5.**

$$\left[ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right] \times$$
$$\left\{ \sum_{\alpha=1}^{l_u-1} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k] \right] \right\}$$
$$= \left[ \sum_{h=0,\ k=1}^{s-m,\ t} (-1)^{h+k} S_{1m}(b^{'}-c^{'}) S_{h(s-m)}(c^{''}-b^{''}) S_{kt}(d) \frac{k}{(h+k)\ u^{h+k}} \right] q^{s+t-1}$$
$$+ \ (\ terms \ \ of \ \ degree \ \leq \ s+t-2 \ \ in \ \ q \ \ over\ Q\,[\epsilon]\ ),$$

*where* $c^{''}-b^{''}=(c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $d=(d_1,\ldots,d_t)$, $S_{0(s-m)}(c^{''}-b^{''}):=1$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$ *.*

**Proof :** By applying Lemma 2.6 and Lemma 3.4, this lemma can be proved. □

**Lemma 3.6.**

$$\left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times \left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times$$
$$\left\{ \prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]_+ \right\} \qquad (\triangle\triangle\triangle\triangle)$$
$$= \ terms \ \ of \ \ degree \ \leq \ s+t-2 \ \ in \ \ q \ \ over \ \ Q\,[\epsilon]\,, \ for \ \ q \gg 0.$$

**Proof :** We prove this lemma by discussing on $u$.

*Case 1 :* Suppose $u=d_1$, that is, $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+=0$, and $l_u=\frac{q-v-\epsilon}{d_1}$ .

It follows that for $q \gg 0$, the term $(\triangle\triangle\triangle\triangle)$ is equal to

$$\left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times \left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-l_u d_k) \right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{d_1}$ , we obtain the following expression for $(\triangle\triangle\triangle\triangle)$ :

$$\frac{v+\epsilon}{{d_1}^{s-m+t-1}} \left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [(d_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-d_1 b_h] \right\} \times \left\{ \prod_{k=2}^{t} [(d_1-d_k)q+(v+\epsilon)d_k] \right\},$$

which is a polynomial of degree less than $s+t-1$ in $q$ over $Q\,[\epsilon]$.

*Case 2* $(m \neq 0)$ *:* Suppose $u = b_1 - c_1$, that is, $l_u = \frac{q-v-\epsilon}{b_1-c_1}$ .
If $b_1 - c_1 = d_1$, then $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+ = 0$, and $(\triangle\triangle\triangle\triangle)$ is equal to

$$\left\{\prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_u d_k)\right\}.$$

Similarly, the term $(\triangle\triangle\triangle\triangle)$ can be expressed as a polynomial of degree less than $s+t-1$ in $q$ over $Q\,[\epsilon]$ for $q \gg 0$.
If $b_1 - c_1 > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(b_1-1-2c_1)d_k}{(b_1-c_1)-d_k}$ .
It follows that for $q \gg 0$, $(\triangle\triangle\triangle\triangle)$ has the form

$$\left\{\prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]\right\}.$$

Replacing $l_u$ with $\frac{q-v-\epsilon}{b_1-c_1}$ , we obtain the following expression for $(\triangle\triangle\triangle\triangle)$ :

$$\frac{1}{(b_1-c_1)^{\,s-m+t}}\left\{\prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[(b_1-c_1-(c_h-b_h))q+(v+\epsilon)(c_h-b_h)-(b_1-c_1)b_h]\right\} \times \left\{\prod_{k=1}^{t}[(b_1-c_1-d_k)q+(v+\epsilon)d_k] - \prod_{k=1}^{t}[(b_1-c_1-d_k)q+(v+\epsilon-(b_1-c_1))d_k]\right\}.$$

Therefore, the term $(\triangle\triangle\triangle\triangle)$ can be expressed as a polynomial of degree less than $s+t-1$ in $q$ over $Q\,[\epsilon]$, for $q \gg 0$.

*Case 3* $(m \neq s)$ *:* Suppose $u = c_{m+1} - b_{m+1}$, that is, $l_u = \frac{q-v-\epsilon}{c_{m+1}-b_{m+1}}$ .
If $c_{m+1} - b_{m+1} = d_1$, then $\prod_{k=1}^{t}[q-(l_u+1)d_k]_+ = 0$, and $(\triangle\triangle\triangle\triangle)$ is equal to

$$\left\{\prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j)\right\} \times \left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\} \times \left\{\prod_{k=1}^{t}(q-l_u d_k)\right\}.$$

Similarly, the term $(\triangle\triangle\triangle\triangle)$ can be expressed as a polynomial of degree less than $s+t-1$ in $q$ over $Q\,[\epsilon]$ for $q \gg 0$.
If $c_{m+1} - b_{m+1} > d_1$, then $q-(l_u+1)d_k > 0$ for $q > \frac{(c_{m+1}-1-2b_{m+1})d_k}{(c_{m+1}-b_{m+1})-d_k}$ .

It follows that for $q \gg 0$, the term $(\triangle\triangle\triangle\triangle)$ is equal to

$$\left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times \left\{ \prod_{h=m+1}^{s} [q - l_u(c_h - b_h) - b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q - l_u d_k) - \prod_{k=1}^{t} [q - (l_u+1) d_k] \right\}.$$

Similarly, the term $(\triangle\triangle\triangle\triangle)$ can be expressed as a polynomial of degree less than $s+t-1$ in $q$ over $Q[\epsilon]$ for $q \gg 0$. $\square$

**Lemma 3.7.**

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)] - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)] \right\} \times \left\{ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)] \right\} \times \left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k] \right\}$$
$$= \left[ \sum_{j=1,\ h=0,\ k=1}^{m,\ s-m,\ t} (-1)^{j+h+k} S_{jm}(b^{'}-c^{'}) S_{h(s-m)}(c^{''}-b^{''}) S_{kt}(d) \frac{j\ k}{(j+h+k-1)\ u^{j+h+k-1}} \right] q^{s+t-1}$$
$$+ \ (\ \textit{terms of degree} \le s+t-2 \ \textit{in} \ q \ \textit{over} \ Q[\epsilon]\ ),$$

*where* $b^{'}-c^{'} = (b_1-c_1,\ldots,b_m-c_m)$, $c^{''}-b^{''} = (c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $d=(d_1,\ldots,d_t)$, $S_{0(s-m)}(c^{''}-b^{''}) := 1$, *and* $Q[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** The proof is analogous to Lemma 2.9 . $\square$

**Lemma 3.8.**

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)] - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)] \right\} \times \left\{ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k] \right\}$$
$$= \left[ \sum_{j=1,\ h=0,\ k=1}^{m,\ s-m,\ t} (-1)^{j+h+k} S_{jm}(b^{'}-c^{'}) S_{h(s-m)}(c^{''}-b^{''}) S_{kt}(d) \frac{j\ k}{(j+h+k-1)\ u^{j+h+k-1}} \right] q^{s+t-1}$$
$$+ \ (\ \textit{terms of degree} \le s+t-2 \ \textit{in} \ q \ \textit{over} \ Q[\epsilon]\ ),$$

*where* $b^{'}-c^{'} = (b_1-c_1,\ldots,b_m-c_m)$, $c^{''}-b^{''} = (c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $d=(d_1,\ldots,d_t)$, $S_{0(s-m)}(c^{''}-b^{''}) := 1$, *and* $Q[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** Since

$$\prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h]$$
$$= \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)] - \sum_{m+1\ \le\ h_1\ \le\ s} \left[ b_{h_1} \prod_{h\neq h_1} [q-\alpha(c_h-b_h)] \right]$$
$$+ \sum_{m+1\ \le\ h_1\ <\ h_2\ \le\ s} \left[ b_{h_1} b_{h_2} \prod_{h\neq h_1,\ h_2} [q-\alpha(c_h-b_h)] \right] - \cdots + (-1)^{s-m} b_{m+1}\cdots b_s,$$

by applying Lemma 3.7, this lemma can be proved. □

**Lemma 3.9.**

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j\ ] - \prod_{j=1}^{m} [q-(\alpha+1)(b_j-c_j)-c_j\ ] \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t} (q-\alpha d_k) - \prod_{k=1}^{t} [q-(\alpha+1)d_k] \right\}$$
$$= \left[ \sum_{j=1,\ h=0,\ k=1}^{m,\ s-m,\ t} (-1)^{j+h+k} S_{jm}(b^{'}-c^{'}) S_{h(s-m)}(c^{''}-b^{''}) S_{kt}(d) \frac{j\ k}{(j+h+k-1)\ u^{j+h+k-1}} \right] q^{s+t-1}$$
$$+\ (\ \textit{terms of degree} \ \le\ s+t-2 \ \textit{in}\ q\ \textit{over}\ Q\,[\epsilon]\ ),$$

*where* $b^{'}-c^{'}=(b_1-c_1,\ldots,b_m-c_m)$, $c^{''}-b^{''}=(c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $d=(d_1,\ldots,d_t)$, $S_{0(s-m)}(c^{''}-b^{''}):=1$, *and* $Q\,[\epsilon]$ *is the polynomial ring in* $\epsilon$ *over* $Q$.

**Proof :** Since

$$\prod_{j=1}^{m} [q-\alpha(b_j-c_j)-c_j\ ]$$
$$= \prod_{j=1}^{m} [q-\alpha(b_j-c_j)] - \sum_{1\ \le\ j_1\ \le\ m} \left[ c_{j_1} \prod_{j\ \neq\ j_1} [q-\alpha(b_j-c_j)] \right]$$
$$+ \sum_{1\ \le\ j_1\ <\ j_2\ \le\ m} \left[ c_{j_1} c_{j_2} \prod_{j\ \neq\ j_1,\ j_2} [q-\alpha(b_j-c_j)] \right] - \cdots + (-1)^m c_1\cdots c_m, \quad \text{and}$$

$$\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ]$$
$$=\prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)]-\sum_{1\ \leq\ j_1\ \leq\ m}\left[c_{j_1}\prod_{j\ \neq\ j_1}[q-(\alpha+1)(b_j-c_j)]\right]$$
$$+\sum_{1\ \leq\ j_1\ <\ j_2\ \leq\ m}\left[c_{j_1}c_{j_2}\prod_{j\ \neq\ j_1,\ j_2}[q-(\alpha+1)(b_j-c_j)]\right]-\cdots+(-1)^m c_1\cdots c_m,$$

by applying Lemma 3.8, this lemma can be proved. □

**Lemma 3.10.**

$$\left\{\prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j\ ]-\prod_{j=1}^{m}[q-(l_u+1)(b_j-c_j)-c_j\ ]_+\right\}\times$$
$$\left\{\prod_{h=m+1}^{s}[q-l_u(c_h-b_h)-b_h]\right\}\times\left\{\prod_{k=1}^{t}(q-l_u d_k)-\prod_{k=1}^{t}[q-(l_u+1)d_k]_+\right\}$$
$$=\textit{terms of degree}\ \leq\ s+t-2\ \textit{ in } q\ \textit{ over }\ Q\,[\epsilon]\,,\ \textit{for}\ q\gg 0.$$

**Proof :** The proof is analogous to Lemma 2.12 . □

**Theorem 3.11.** *The Hilbert-Kunz function of the hypersurface*

$$Y_1^{b_1}\dots Y_s^{b_s}+Y_1^{c_1}\dots Y_s^{c_s}Z_1^{d_1}\dots Z_t^{d_t}$$

*is*

$$n\ \longmapsto\ \lambda p^{\ (s+t-1)n}\ +\ \sum_{k=0}^{s+t-2}f_k\,(n)\,p^{\ kn}\qquad \textit{for}\ \ n\gg 0,$$

*where* $\lambda=S_{1s}(b)+$

$$\left[\sum_{j=2,\ h=0,\ k=1}^{m,\ s-m,\ t}(-1)^{j+h+k}S_{jm}(b^{'}-c^{'})S_{h(s-m)}(c^{''}-b^{''})S_{kt}(d)\frac{j\ k}{(j+h+k-1)u^{j+h+k-1}}\right]$$

$b=(b_1,\dots,b_s)$, $d=(d_1,\dots,d_t)$, $b^{'}-c^{'}=(b_1-c_1,\dots,b_m-c_m)$, $c^{''}-b^{''}=(c_{m+1}-b_{m+1},\dots,c_s-b_s)$, $0\leq m\leq s$, *and* $f_k(n)$ *is an eventually periodic function of* $n$ *for each* $k$.

**Proof :** Let $q = p^n$. By Proposition 3.2, $HK_R(q)$ is equal to the sum of five terms :

$$q^{s+t} - q^t \prod_{j=1}^{s}(q-b_j), \tag{1}$$

$$\left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times$$
$$\left\{ \sum_{\alpha=1}^{l_u-1} \left[ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right] \times \left[ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k] \right] \right\}, \tag{2}$$

$$\left\{ \prod_{j=1}^{m}(q-b_j) - \prod_{j=1}^{m}(q-c_j) \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]_+ \right\}, \tag{3}$$

$$\sum_{\alpha=1}^{l_u-1} \left\{ \prod_{j=1}^{m}[q-\alpha(b_j-c_j)-c_j\ ] - \prod_{j=1}^{m}[q-(\alpha+1)(b_j-c_j)-c_j\ ] \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [q-\alpha(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-\alpha d_k) - \prod_{k=1}^{t}[q-(\alpha+1)d_k] \right\}, \tag{4}$$

$$\left\{ \prod_{j=1}^{m}[q-l_u(b_j-c_j)-c_j\ ] - \prod_{j=1}^{m}[q-(l_u+1)(b_j-c_j)-c_j\ ]_+ \right\} \times$$
$$\left\{ \prod_{h=m+1}^{s} [q-l_u(c_h-b_h)-b_h] \right\} \times \left\{ \prod_{k=1}^{t}(q-l_u d_k) - \prod_{k=1}^{t}[q-(l_u+1)d_k]_+ \right\}. \tag{5}$$

By applying Lemma 3.3 , Lemma 3.5 , Lemma 3.6 , Lemma 3.9 , and Lemma 3.10 to the five terms, we have

$$HK_R(q) = \lambda q^{s+t-1} + \triangle_{s+t-2}(\epsilon)q^{s+t-2} + \cdots + \triangle_1(\epsilon)q + \triangle_0(\epsilon),$$

where $\lambda = S_{1s}(b) +$
$$\left[ \sum_{j=2,\ h=0,\ k=1}^{m,\ s-m,\ t} (-1)^{j+h+k} S_{jm}(b^{'}-c^{'}) S_{h(s-m)}(c^{''}-b^{''}) S_{kt}(d) \frac{j\ k}{(j+h+k-1)u^{j+h+k-1}} \right]$$

$b^{'}-c^{'} = (b_1-c_1,\ldots,b_m-c_m)$, $c^{''}-b^{''} = (c_{m+1}-b_{m+1},\ldots,c_s-b_s)$, $0 \le m \le s$, and $\triangle_{s+t-2}(\epsilon)$, $\ldots$, $\triangle_1(\epsilon)$, and $\triangle_0(\epsilon)$ are polynomials in $\epsilon$ over $Q$.

Let $f_k(n) := \triangle_k(\epsilon), \quad k = 0,1,2,\ldots,s+t-2.$

Since $\epsilon$ is the remainder of $q-v$ divided by $u$, $f_k(n)$ is an eventually periodic function of $n$ for $n \gg 0$. □